MATHEMATICS I
FIRST SEMESTER

Lec. 04 Matrices and Vectors

Outlines

- Matrices
- Type of Matrix
- Operations on Matrices
- Vector
- Operations on Vectors

Matrices:

A **matrix** is a group of numbers(elements) that are arranged in rows and columns. In general, an $m \times n$ matrix is a rectangular array of mn numbers (or elements) arranged in m rows and n columns. If $m = n$ the matrix is called a square matrix. For example a 2×2 matrix is

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

and an 3×3 matrix is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}_{n\times m} = [a_{ij}], \quad i = 1,2,\ldots,n \quad , \quad j = 1,2,\ldots,m.$$

Generally, we use bold phase letter, like $\boldsymbol{A}$, to denote a matrix, and lower case letters with subscripts, like $\boldsymbol{a_{ij}}$, to denote element of a matrix. Here $\boldsymbol{a_{ij}}$ would be the element at $\boldsymbol{i^{th}}$ row and $\boldsymbol{j^{th}}$ column. So $\boldsymbol{a_{11}}$ is an element at $\boldsymbol{1^{st}}$ row and column. Sometime we use the abbreviation $\boldsymbol{A} = (\boldsymbol{a_{ij}})$ for a matrix with elements $\boldsymbol{a_{ij}}$.

Types of matrices:

1) Square Matrix: It is a matrix whose number of rows are equal to the number of columns (n=m).For example:

$$A = \begin{bmatrix} 2 & 7 \\ 3 & 6 \end{bmatrix}_{2\times 2}, \quad B = \begin{bmatrix} 2 & 4 & 1 \\ 4 & 3 & 2 \\ 2 & 9 & 1 \end{bmatrix}_{3\times 3}$$

2) Null or Zero Matrix: A matrix each of whose elements is zero is called null matrix or zero matrix, for example $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{3\times 2}$ is a (3×2) null matrix.

3) Diagonal Matrix: It is a square matrix which all its elements are zero except the elements on the main diagonal. For example:

$$A=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

4) Identity Matrix: It is a diagonal matrix whose elements on the main diagonal are equal to 1, and it is denoted by I_n. For example:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

5)Transpose Matrix: Transpose of A is denoted by (A^T), means that write the rows of A as columns in A^t. For example:

$$A = \begin{bmatrix} 5 & 4 \\ 7 & -2 \\ -8 & 1 \end{bmatrix}_{3\times2}, \qquad A^T = \begin{bmatrix} 5 & 7 & -8 \\ 4 & -2 & 1 \end{bmatrix}_{2\times3}$$

6) Symmetric Matrix: A square matrix A such that $A = A^T$ is called symmetric matrix i.e. A is a symmetric matrix if and only if aij = a_{ji} for all element.

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}_{3\times 3}$$

Operations on Matrices:

(i) **Equality:** *Two matrix* $\boldsymbol{A} = (\boldsymbol{a_{ij}})$ *and* $\boldsymbol{B} = (\boldsymbol{b_{ij}})$ *are equal if corresponding elements are equal, i.e.* $\boldsymbol{a_{ij}} = \boldsymbol{b_{ij}}$.

(ii) **Addition:** *If* $\boldsymbol{A} = (\boldsymbol{a_{ij}})$ *and* $\boldsymbol{B} = (\boldsymbol{b_{ij}})$ *and the sum of* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *is* $\boldsymbol{A} + \boldsymbol{B} = (\boldsymbol{c_{ij}}) = \boldsymbol{a_{ij}} + \boldsymbol{b_{ij}}$.

(iii) **Scalar Product:** *If* $\boldsymbol{A} = (\boldsymbol{a_{ij}})$ *is matrix and* $\boldsymbol{k}$ *is number(scalar), the* $\boldsymbol{kA} = (\boldsymbol{ka_{ij}})$ *is product of* $\boldsymbol{k}$ *and* $\boldsymbol{A}$.

From the abovewe see that, to multiply a matrix by a number $\boldsymbol{k}$, we simply multiply each of its entries by $\boldsymbol{k}$; to add two matrices we just add their corresponding entries; $\boldsymbol{A} - \boldsymbol{B} = \boldsymbol{A} + (-1)\boldsymbol{B}$.

Example 1. *Let* $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$ *and* $B = \begin{bmatrix} 0 & 5 \\ 3 & -4 \end{bmatrix}$,

find : (a) A + B, (b) 3A, (c) 4A - B:

Solution

(a)

$$A + B = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 5 \\ 3 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 2+0 & 3+5 \\ -1+3 & 4+(-4) \end{bmatrix} = \begin{bmatrix} 2 & 8 \\ 2 & 0 \end{bmatrix}$$

(b)

$$3A = 3\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ -3 & 12 \end{bmatrix}$$

(c)

$$4A - B = \begin{bmatrix} 8 & 12 \\ -4 & 16 \end{bmatrix} - \begin{bmatrix} 0 & 5 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 8 & 7 \\ -7 & 20 \end{bmatrix}$$

The following fact lists all properties of matrix addition and scalar product.

THEOREM 1.1. *Let* $\boldsymbol{A}$, $\boldsymbol{bB}$, *and* $\boldsymbol{C}$ *be matrices. Let* $\boldsymbol{a}, \boldsymbol{b}$ *be scalars (numbers). We have*

(1) $\boldsymbol{A} + \boldsymbol{0} = \boldsymbol{0} + \boldsymbol{A} = \boldsymbol{A}$, $\boldsymbol{A} - \boldsymbol{A} = \boldsymbol{0}$;
(2) $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$ *(commutativity);*
(3) $\boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C}$, $(\boldsymbol{ab})\boldsymbol{A} = \boldsymbol{a}(\boldsymbol{bA})$ *(associativity);*
(4) $\boldsymbol{a}(\boldsymbol{A}+\boldsymbol{B}) = \boldsymbol{aA}+\boldsymbol{aB}$, $(\boldsymbol{a}+\boldsymbol{b})\boldsymbol{A} = \boldsymbol{aA}+\boldsymbol{bA}$ *(distributivity)*

Matrices multiplication

Let A be an n× k matrix and B be a k×m matrix then C=AB is an n×m matrix, where the element in the i^{th} row and j^{th} column of AB is the sum

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}, \quad i = 1,2,\ldots,m \ and \ j = 1,2,\ldots,p.$$

Example 2: Let $A = \begin{pmatrix} 2 & 3 \\ -1 & 4 \\ 5 & -2 \end{pmatrix}_{3\times 2}$ *and* $B = \begin{pmatrix} 3 & 1 & 4 & -5 \\ -2 & 0 & 3 & 4 \end{pmatrix}$

$$AB = \begin{pmatrix} 0 & 2 & 17 & 2 \\ -11 & -1 & 8 & 21 \\ 19 & 5 & 14 & -33 \end{pmatrix}_{3\times 4}$$

Example 3: *Let* $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$ *and* $B = \begin{bmatrix} 0 & 5 \\ 3 & -4 \end{bmatrix}$, *find AB*

Solution

$$AB = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 0 & 5 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 9 & -2 \\ 12 & -21 \end{bmatrix}$$

The following fact gives properties of matrix product,

Theorem 1.2. *Let A;B;C be three matrices and r be a scalar, we have*

- $A(BC) = (AB)C$, $r(AB) = A(rB)$ *(associativity)*
- $A(B + C) = AB + AC$ *(distributivity)*

Notice, in general $AB \neq BA$, that is for most of the times, AB is not equal to BA.

Determinants

With each square matrix A we associate a number det(A) or $|a_{ij}|$ called the determinant of A, calculated from the entries of A as follows:

For n=1, det(a)=a,

For n=1, det(a)=a, $\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{21}a_{12}$

For n =3,

(i) the determinant of matrix is defined as follows:

$$\begin{vmatrix} + & - & + \\ a11 & a22 & a23 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{vmatrix} = a11 \begin{vmatrix} a22 & a23 \\ a32 & a33 \end{vmatrix} - a12 \begin{vmatrix} a21 & a23 \\ a31 & a33 \end{vmatrix} + a13 \begin{vmatrix} a21 & a22 \\ a31 & a32 \end{vmatrix}$$

(ii) Consider the (3×3) matrix $\begin{vmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{vmatrix}$

$= \text{a11 a22 a33} + \text{a21 a22 a31} + \text{a13 a21 a32}$

$-$ a31 a22 a13 - a32 a23 a11 -a33 a21 a12.

where the first two columns are rewritten to the right of the matrix.

Example 3: Find the determinant of each matrix

a) $\begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$

$$\begin{vmatrix} 1 & 3 \\ -2 & 5 \end{vmatrix} = 11$$

b) $\begin{bmatrix} 2 & 4 \\ 6 & 12 \end{bmatrix}$

$$\begin{vmatrix} 2 & 4 \\ 6 & 12 \end{vmatrix} = 0$$

Example 2: Find the determinant of A where:

$$A = \begin{bmatrix} 1 & 3 & -5 \\ -2 & 4 & 6 \\ 0 & -7 & 9 \end{bmatrix}$$

Sol.: By choosing the first column we get

$$\det(A) = \begin{vmatrix} 1 & 3 & -5 \\ -2 & 4 & 6 \\ 0 & -7 & 9 \end{vmatrix} = 1 \cdot \begin{vmatrix} 4 & 6 \\ -7 & 9 \end{vmatrix} - (-2) \cdot \begin{vmatrix} 3 & -5 \\ -7 & 9 \end{vmatrix} + 0 \cdot \begin{vmatrix} 3 & -5 \\ 4 & 6 \end{vmatrix}$$

$$= 62$$

Vector:

A vector is a matrix that has only one row – then we call the matrix a row vector or only one column then we call it a column vector. A row vector is of the form: $a = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$

A column vector is of the form: $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

Transpose of a Vector: Sometimes it is helpful to deal with a row version of a vector, which is called the transpose of the vector and is denoted with a superscript T:

$$\mathbf{v}^T = \begin{bmatrix} v_1 & \cdots & v_N \end{bmatrix}$$

We can also use the following variation of this:

$$\mathbf{v} = \begin{bmatrix} v_1 & \cdots & v_N \end{bmatrix}^T$$

Operations on Vectors

1) Vector Addition:

$$\mathbf{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_N \end{bmatrix} \qquad \mathbf{a} + \mathbf{b} = \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_N + b_N \end{bmatrix}$$

2) Multiply by a Scalar:

$$\mathbf{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix} \qquad \alpha\mathbf{a} = \begin{bmatrix} \alpha a_1 \\ \vdots \\ \alpha a_N \end{bmatrix}$$

Note: multiplying a vector by a scalar is viewed as changing its length. If the scalars are real: (i) multiplying by a scalar whose magnitude is greater than 1 increases the length; (ii) multiplying by a scalar whose magnitude is less than 1 decreases the length; (iii) if the scalar is negative, multiplying by it “flips” the vector to point in the opposite direction

Example 4:

Vector Addition. Multiplication by Scalars

With respect to a given coordinate system, let

$$\mathbf{a} = [4, 0, 1] \quad \text{and} \quad \mathbf{b} = [2, -5, \tfrac{1}{3}].$$

Then $-\mathbf{a} = [-4, 0, -1]$, $7\mathbf{a} = [28, 0, 7]$, $\mathbf{a} + \mathbf{b} = [6, -5, \tfrac{4}{3}]$, and

$$2(\mathbf{a} - \mathbf{b}) = 2[2, 5, \tfrac{2}{3}] = [4, 10, \tfrac{4}{3}] = 2\mathbf{a} - 2\mathbf{b}.$$

Properties of Vectors:

Let **x**, **y**, and **z** be vectors of the same dimension and let α and β be scalars; then the following properties hold:

(1) Commutativity	$\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ $\alpha\mathbf{x} = \mathbf{x}\alpha$
(2) Associativity	$(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{y} + (\mathbf{x} + \mathbf{z})$ $\alpha(\beta\mathbf{x}) = (\alpha\beta)\mathbf{x}$
(3) Distributivity	$\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$ $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$

Example 5: let x= $\begin{bmatrix}3\\5\\7\end{bmatrix}$,y= $\begin{bmatrix}2\\1\\3\end{bmatrix}$ and z= $\begin{bmatrix}4\\6\\8\end{bmatrix}$. Find

1) $x + y = y + x$

$$x + y = \begin{bmatrix}3\\5\\7\end{bmatrix} + \begin{bmatrix}2\\1\\3\end{bmatrix} = \begin{bmatrix}5\\6\\10\end{bmatrix}$$

$$y + x = \begin{bmatrix}2\\1\\3\end{bmatrix} + \begin{bmatrix}3\\5\\7\end{bmatrix} = \begin{bmatrix}5\\6\\10\end{bmatrix}$$

1) $(x + y) + z = \left(\begin{bmatrix}3\\5\\7\end{bmatrix} + \begin{bmatrix}2\\1\\3\end{bmatrix}\right) + \begin{bmatrix}4\\6\\8\end{bmatrix} = \begin{bmatrix}5\\6\\10\end{bmatrix} + \begin{bmatrix}4\\6\\8\end{bmatrix} = \begin{bmatrix}9\\12\\18\end{bmatrix}$

$$x + (y + z) = \begin{bmatrix}3\\5\\7\end{bmatrix} + \left(\begin{bmatrix}2\\1\\3\end{bmatrix} + \begin{bmatrix}4\\6\\8\end{bmatrix}\right) = \begin{bmatrix}3\\5\\7\end{bmatrix} + \begin{bmatrix}6\\7\\11\end{bmatrix} = \begin{bmatrix}9\\12\\18\end{bmatrix}$$

EXERCISES 4.1:

1) Let

$$\mathbf{A} = \begin{bmatrix} 0 & 2 & 4 \\ 6 & 5 & 5 \\ 1 & 0 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 5 & 2 \\ 5 & 3 & 4 \\ -2 & 4 & -2 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 5 & 2 \\ -2 & 4 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} -4 & 1 \\ 5 & 0 \\ 2 & -1 \end{bmatrix},$$

$$\mathbf{E} = \begin{bmatrix} 0 & 2 \\ 3 & 4 \\ 3 & -1 \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} 1.5 \\ 0 \\ -3.0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -5 \\ -30 \\ 10 \end{bmatrix}.$$

Find:

1) $2\mathbf{A} + 4\mathbf{B}, \quad 4\mathbf{B} + 2\mathbf{A}, \quad 0\mathbf{A} + \mathbf{B}, \quad 0.4\mathbf{B} - 4.2\mathbf{A}$

2) $(\mathbf{C} + \mathbf{D}) + \mathbf{E}, \quad (\mathbf{D} + \mathbf{E}) + \mathbf{C}, \quad 0(\mathbf{C} - \mathbf{E}) + 4\mathbf{D}.$

3) $(5\mathbf{u} + 5\mathbf{v}) - \frac{1}{2}\mathbf{w}, \quad -20(\mathbf{u} + \mathbf{v}) + 2\mathbf{w},$

2) Let

$$\mathbf{A} = \begin{bmatrix} 4 & -2 & 3 \\ -2 & 1 & 6 \\ 1 & 2 & 2 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \\ -2 & 0 \end{bmatrix}, \qquad \mathbf{a} = [1 \quad -2 \quad 0], \quad \mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}.$$

Find:

1) $\mathbf{AB}, \quad \mathbf{AB}^T, \quad \mathbf{BA}, \quad \mathbf{B}^T\mathbf{A}$

2) $\mathbf{Aa}, \quad \mathbf{Aa}^T, \quad (\mathbf{Ab})^T, \quad \mathbf{b}^T\mathbf{A}^T$

3) $3\mathbf{A} - 2\mathbf{B}, \quad (3\mathbf{A} - 2\mathbf{B})^T, \quad 3\mathbf{A}^T - 2\mathbf{B}^T,$

4) $\mathbf{ab}, \quad \mathbf{ba}, \quad 1.5\mathbf{a} + 3.0\mathbf{b}, \quad 1.5\mathbf{a}^T + 3.0\mathbf{b},$