MATHEMATICS I

FIRST SEMESTER

Lec. 03

Trigonometric Functions

Outlines

- Radian Measure
- Types and Trigonometric Relation
- Periodicity and Graphs of the Trigonometric Functions
- Even and Odd Trigonometric Functions
- Identities
  - Addition Formulas
  - Double-Angle Formulas
  - Half-Angle Formulas
- The Law of Consines

Trigonometric Functions

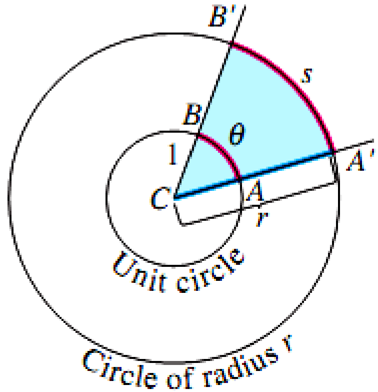
The trigonometric functions are important because they are periodic, or repeating, and therefore model many naturally occurring periodic processes.

•Radian Measure

The **radian measure** of the angle ACB at the center of the unit circle equals the length of the arc that ACB cuts from the unit circle. Figure shows that $s = r\theta$ is the length of arc cut from a circle of radius r when the subtending angle θ

Degrees to radians: multiply by $\frac{\pi}{180}$

Radians to degrees: multiply by $\frac{180}{\pi}$

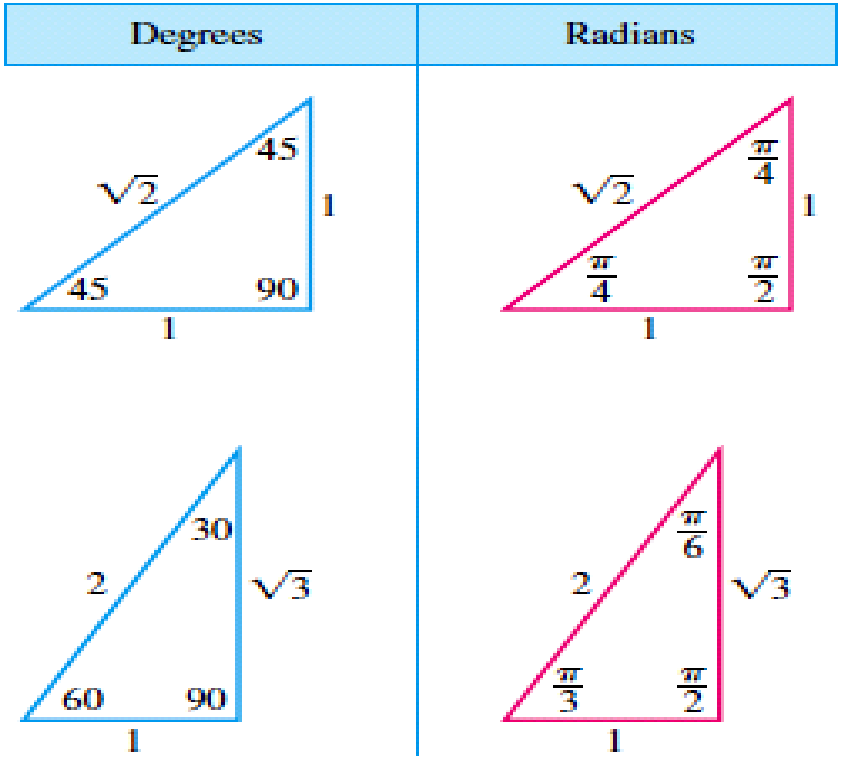


For example:

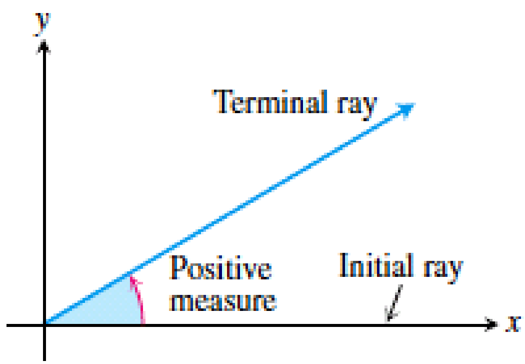
45° in radian measure is $\longrightarrow$ $45 \cdot \frac{\pi}{180} = \frac{\pi}{4}$ rad

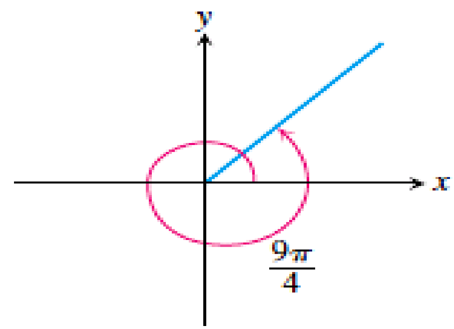
$\frac{\pi}{6}$ in radian measure is $\longrightarrow$ $\frac{\pi}{6} \cdot \frac{180}{\pi} = 30°$

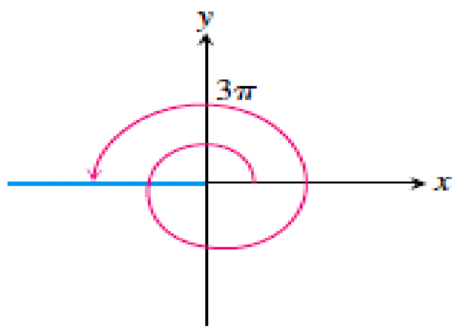
Figure shows the angles of two common triangles in both measures.

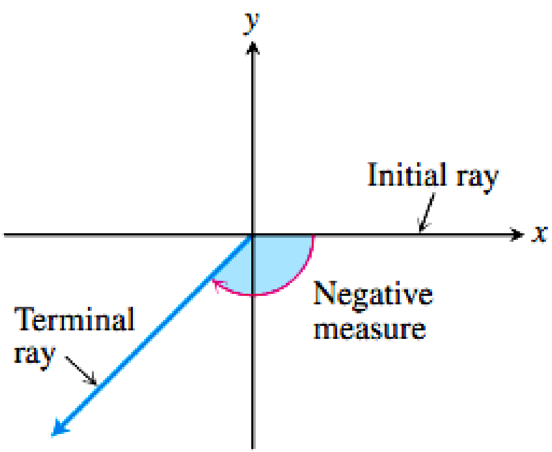


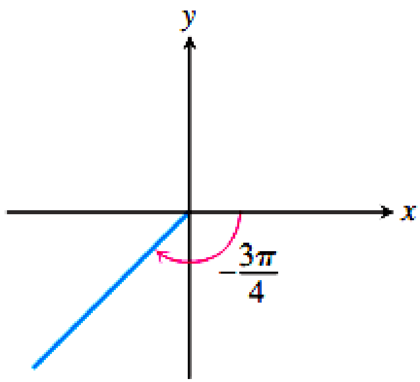
measured counterclockwise from the positive *x*-axis are assigned positive measures; angles measured clockwise are assigned negative measures.

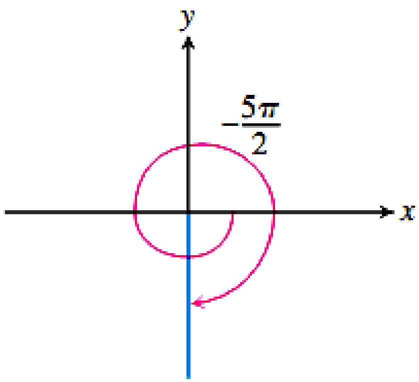






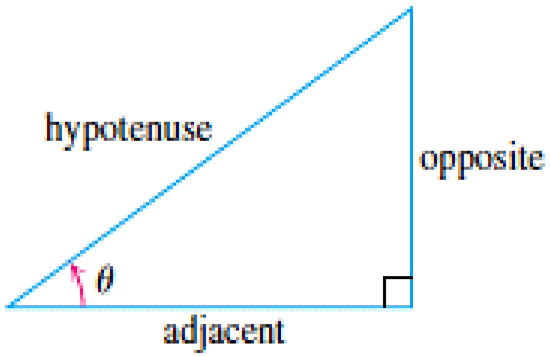






- Types and Trigonometric Relation

The Six Basic Trigonometric Functions



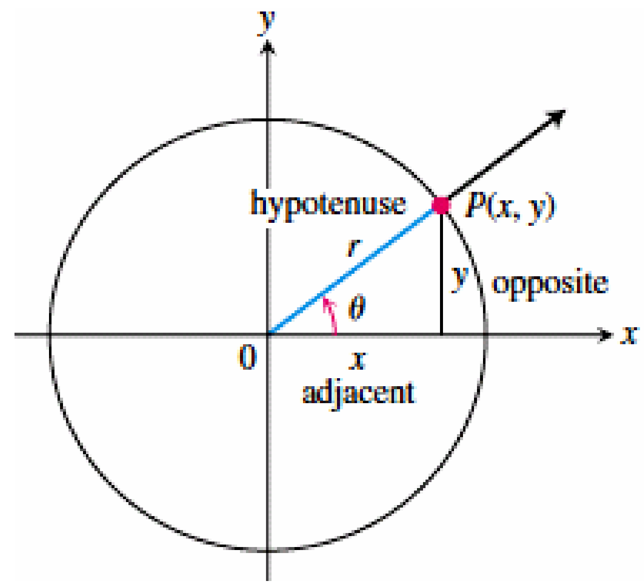
$$\sin\theta = \frac{opp}{hyp} \qquad \cos\theta = \frac{adj}{hyp}$$

$$\tan\theta = \frac{opp}{adj} \qquad \cot\theta = \frac{adj}{opp}$$

$$\sec\theta = \frac{hyp}{adj} \qquad \csc\theta = \frac{hyp}{opp}$$

”the angle in standard position in a circle of radius *r*. We then define the trigonometric functions in terms of the coordinates of the point $P(x, y)$ where the angle's terminal ray intersects the circle (shows in Figure).

- **sine:** $\sin\theta = \frac{y}{r}$ **cosine:** $\cos\theta = \frac{x}{r}$
- **tangent:** $\tan\theta = \frac{y}{x}$ **cotangent:** $\cot\theta = \frac{x}{y}$
- **secant:** $\sec\theta = \frac{r}{x}$ **cosecant:** $\csc\theta = \frac{r}{y}$



Notice also the following definitions, whenever the quotients are defined.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

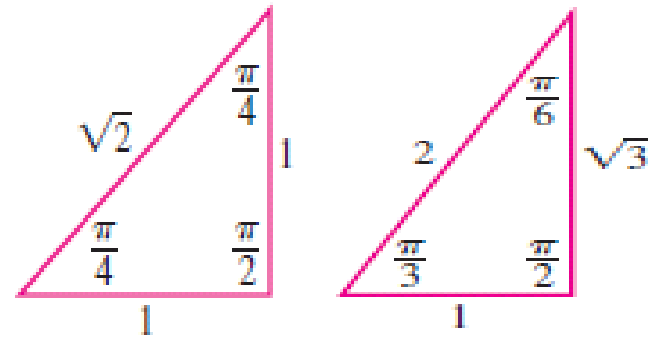
$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

”The exact values of these trigonometric ratios for some angles can be read from the triangles in Figure.

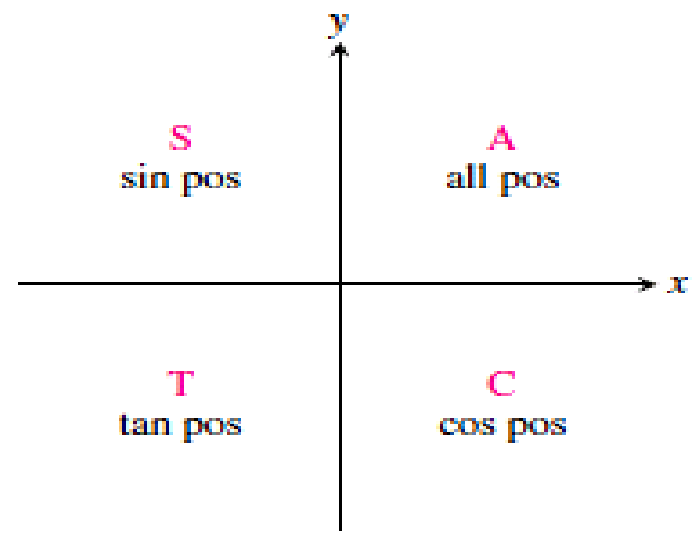
$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ $\quad\sin\frac{\pi}{6} = \frac{1}{2}$ $\quad\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ $\quad\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ $\quad\cos\frac{\pi}{3} = \frac{1}{2}$

$\tan\frac{\pi}{4} = 1$ $\quad\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$ $\quad\tan\frac{\pi}{3} = \sqrt{3}$



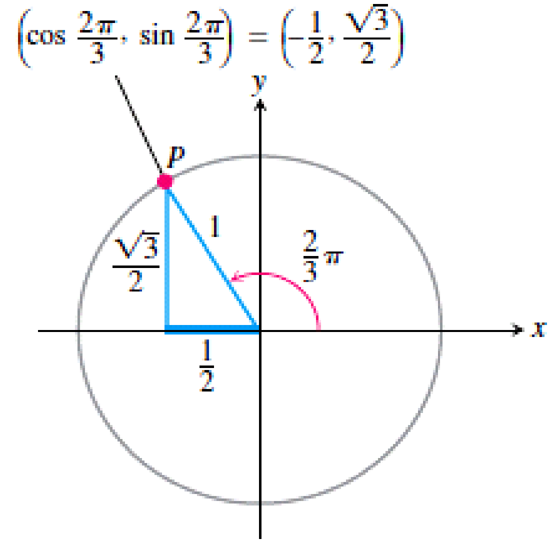
”The CAST rule (shown in Figure) is useful for remembering when the basic trigonometric functions are positive or negative.



" from the triangle in Figure, we see that.

$$\sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2} \qquad \cos \frac{2\pi}{3} = -\frac{1}{2}$$

$$\tan \frac{2\pi}{3} = -\sqrt{3}$$



Using a similar method we determined the values of sin θ, cos θ, and tan θ shown in Table 1.4.

TABLE 1.4 Values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for selected values of θ

Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{-\sqrt{2}}{2}$	-1	$\frac{-\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
$\cos\theta$	-1	$\frac{-\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	-1	0	1
$\tan\theta$	0	1		-1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	-1	$\frac{-\sqrt{3}}{3}$	0		0

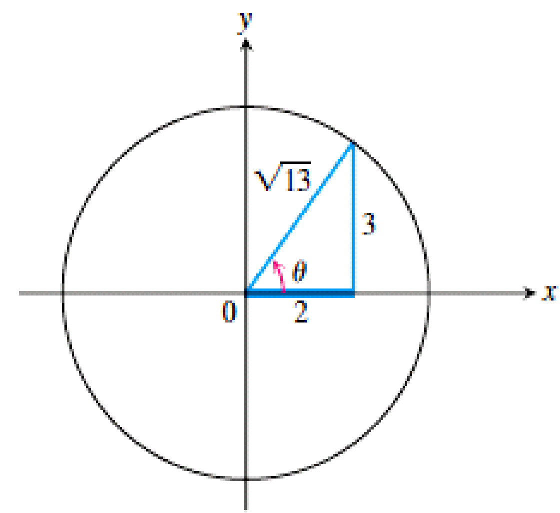
EXAMPLE 1: If $\tan\theta = \frac{3}{2}$ and $0 < \theta < \frac{\pi}{2}$, find the five other trigonometric functions of θ.

Solution : From $\tan\theta = \frac{3}{2}$ we construct the right triangle of height 3 (opposite) and base 2 (adjacent) in Figure . The Pythagorean theorem gives the length of the hypotenuse, $\sqrt{4+9} = \sqrt{13}$. From the triangle we write the values of the other five trigonometric functions:

$$\cos\theta = \frac{2}{\sqrt{13}} \qquad \sin\theta = \frac{3}{\sqrt{13}}$$

$$\sec\theta = \frac{\sqrt{13}}{2} \qquad \csc\theta = \frac{\sqrt{13}}{3}$$

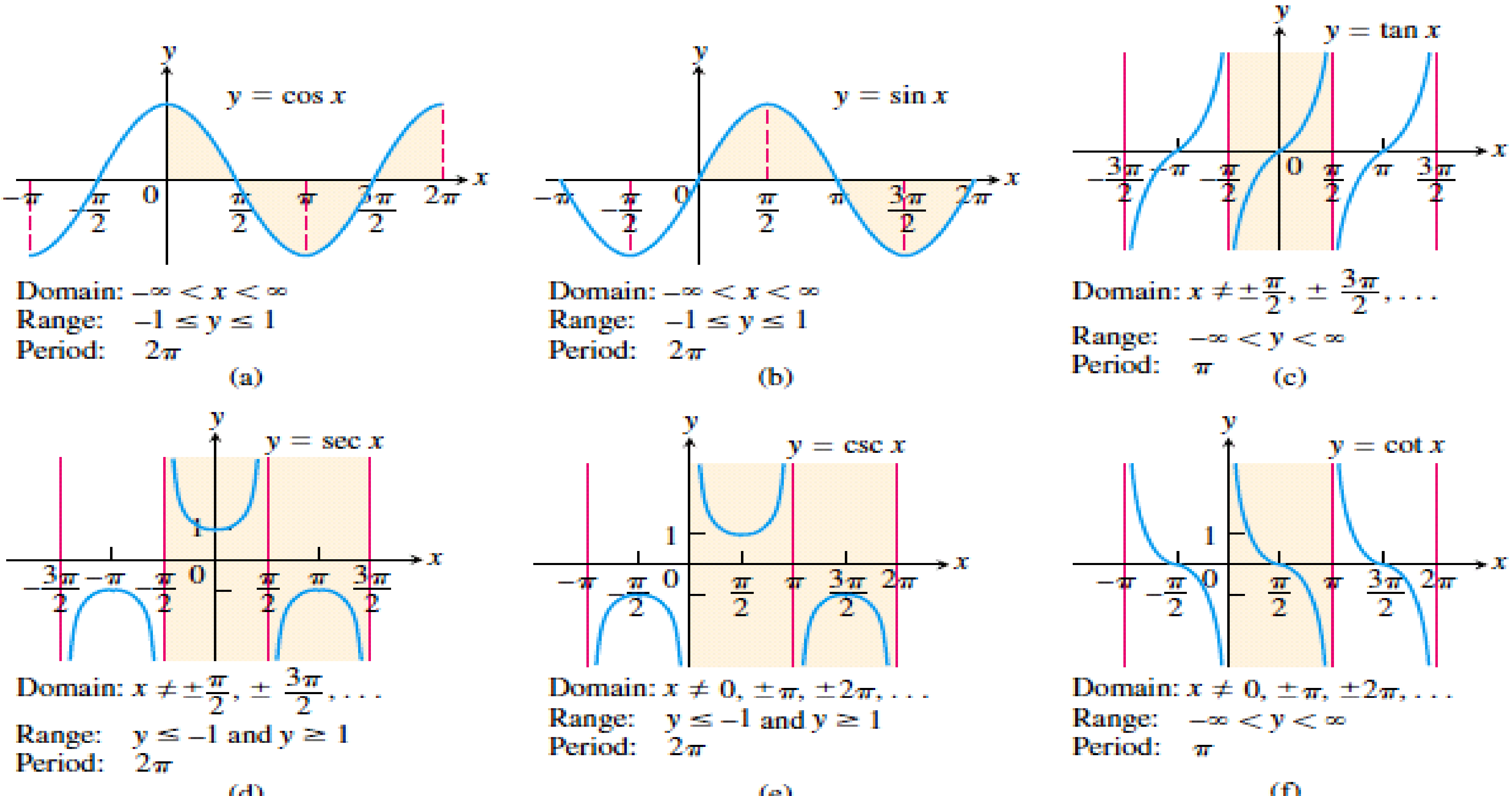
$$\cot\theta = \frac{2}{3}$$



•Periodicity and Graphs of the Trigonometric Functions

Periodic Function: A function f(x) is **periodic** if there is a positive number p such that $f(x+p) = f(x)$ for every value of x. The smallest such value of p is the **period** of f.

”graph trigonometric functions in the coordinate plane: See Figure,



As we can see in Figure the tangent and cotangent functions have period $p = \pi$
The other four functions have period 2π

Even and Odd Trigonometric Functions

The symmetries in the graphs in Figure reveal that the cosine and secant functions are even and the other four functions are odd:

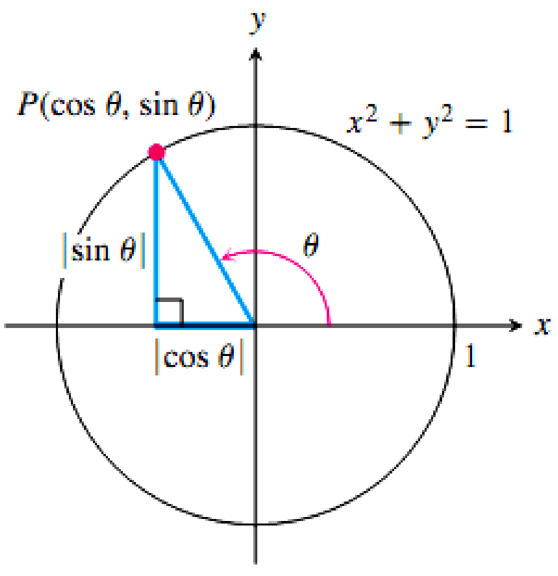
Even	Odd
$\cos(-x) = \cos x$	$\sin(-x) = -\sin x$
$\sec(-x) = \sec x$	$\tan(-x) = -\tan x$
	$\csc(-x) = -\csc x$
	$\cot(-x) = -\cot x$

•Identities

The coordinates of any point P(x, y) in the plane can be expressed in terms of the point's distance from the origin and the angle that ray OP makes with the positive x-axis (shown figure)

Since $x/r = \cos\theta$ and $y/r = \sin\theta$, we have

$$x = r\cos\theta, \qquad y = r\sin\theta.$$



When we can apply the Pythagorean theorem to the reference right triangle in Figure and obtain the equation r = 1

$$\cos^2\theta + \sin^2\theta = 1. \qquad (1)$$

Dividing this identity in turn by $\cos^2\theta$ and $\sin^2\theta$ gives

$$1 + \tan^2\theta = \sec^2\theta.$$
$$1 + \cot^2\theta = \csc^2\theta.$$

Addition Formulas

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

There are similar formulas for

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

Double-Angle Formulas

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$

Additional formulas come from combining the equations

$$\cos^2\theta + \sin^2\theta = 1, \qquad \cos^2\theta - \sin^2\theta = \cos 2\theta.$$

We add the two equations to get $2\cos^2\theta = 1 + \cos 2\theta$ and subtract the second from the first to get $2\sin^2\theta = 1 - \cos 2\theta$. This results in the following identities.

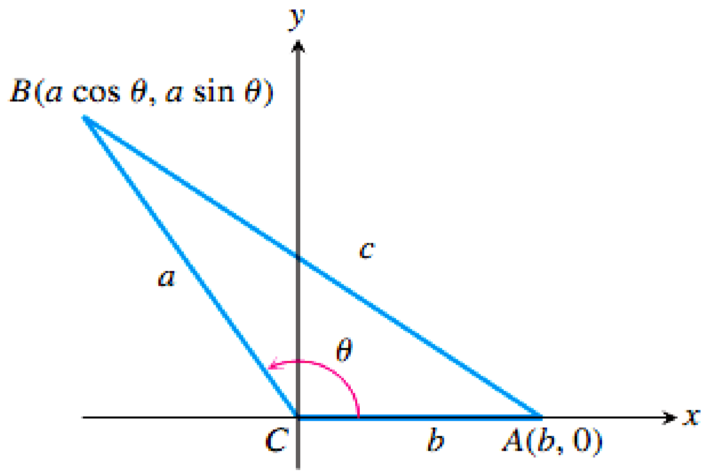
Half-Angle Formulas

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$$
$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

The Law of Cosines

If a, b, and c are sides of a triangle ABC and if θ is the angle opposite c, then

$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$



This equation is called the law of cosines.
as in Figure the coordinates of A

are $(b, 0)$; the coordinates of B are $(a\cos\theta, a\sin\theta)$. The square of the distance between A and B is therefore

$$\begin{aligned} c^2 &= (a\cos\theta - b)^2 + (a\sin\theta)^2 \\ &= a^2\underbrace{(\cos^2\theta + \sin^2\theta)}_{1} + b^2 - 2ab\cos\theta \\ &= a^2 + b^2 - 2ab\cos\theta. \end{aligned}$$

EXERCISES 3.1

1. One of sin *x*, cos *x*, and tan *x* is given. Find the other two if *x* lies in the specified interval. **a.** $\cos x = -\frac{5}{13}$, $x \in [\frac{\pi}{2}, \pi]$ **b.** $\sin x = -\frac{1}{2}$ $x \in [\pi, \frac{3\pi}{2}]$

c. $\tan x = \frac{1}{2}$, $x \in [\pi, \frac{3\pi}{2}]$

2. Graph the following functions. What is the period of each function ?

a. $\cos\frac{\pi x}{2}$ **b.** $-\sin\frac{\pi x}{3}$ **c.** $\cos(x - \frac{\pi}{2})$ d. $\sin(x - \frac{\pi}{4}) + 1$

e. $S = -\tan \pi t$ **h.** $S = \sec(\frac{\pi t}{2})$

3. Use the addition formulas to derive the identities in the following Exercises.

a. $\cos(x - \frac{\pi}{2}) = \sin x$ ***b.*** $\sin(x - \frac{\pi}{2}) = -\cos x$ ***c.*** $\sin(2\pi - x)$

e. Evaluate $\cos\frac{11\pi}{12}$ as $\cos(\frac{\pi}{4} + \frac{2\pi}{3})$ **h.** Evaluate $\sin\frac{5\pi}{12}$

4. Using the Double-Angle Formulas, Find the function values in following Exercises .

a. $cos^2\frac{\pi}{12}$ **b.** $sin^2\frac{\pi}{8}$

5. Derive the formula. $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$, and derive the formula $\tan(A - B)$.

6. What happens if you take $B = A$ in the identity $\cos(A - B) = \cos A \cos B + \sin A \sin B$?