MATHEMATICS I

SECOND SEMESTER

Lec.12 Complex Numbers

Outlines

- Complex Numbers
- Arithmetical operation and properties
- Argand Diagrams
- Euler’s Formula
- Dr Moivre’s Theorem
- Roots

Complex Numbers

A *complex number* is generally written as $a + bi$ where a and b are real numbers and i, called the *imaginary unit*, has the property that $i^2 = -1$. The real numbers a and b are called the *real* and *imaginary parts* of $a + bi$ respectively.

The complex numbers $a + bi$ and $a - bi$ are called *complex conjugates* of each other.

Arithmetical operation and properties

We make the following definitions.

Equality

$a + ib = c + id$ if and only if $a = c$ and $b = d$.

Two complex numbers (a, b) and (c, d) are *equal* if and only if $a = c$ and $b = d$.

Addition

$(a + ib) + (c + id) = (a + c) + i(b + d)$

The *sum* of the two complex numbers (a, b) and (c, d) is the complex number $(a + c, b + d)$.

Multiplication

$(a + ib)(c + id) = (ac - bd) + i(ad + bc)$

The *product* of two complex numbers (a, b) and (c, d) is the complex number $(ac - bd, ad + bc)$.

$c(a + ib) = ac + i(bc)$

The product of a real number c and the complex number (a, b) is the complex number (ac, bc).

Division

$$\frac{c+id}{a+ib} = \frac{(c+id)(a-ib)}{(a+ib)(a-ib)} = \frac{(ac+bd)+i(ad-bc)}{a^2+b^2}.$$

The result is a complex number $x + iy$ with

$$x = \frac{ac+bd}{a^2+b^2}, \qquad y = \frac{ad-bc}{a^2+b^2},$$

and $a^2 + b^2 \neq 0$, since $a + ib = (a, b) \neq (0, 0)$.

EXAMPLE 1 Arithmetic Operations with Complex Numbers

(a) $(2 + 3i) + (6 - 2i) = (2 + 6) + (3 - 2)i = 8 + i$

(b) $(2 + 3i) - (6 - 2i) = (2 - 6) + (3 - (-2))i = -4 + 5i$

(c) $(2 + 3i)(6 - 2i) = (2)(6) + (2)(-2i) + (3i)(6) + (3i)(-2i)$

$$= 12 - 4i + 18i - 6i^2 = 12 + 14i + 6 = 18 + 14i$$

(d)
$$\frac{2+3i}{6-2i} = \frac{2+3i}{6-2i}\,\frac{6+2i}{6+2i} = \frac{12+4i+18i+6i^2}{36+12i-12i-4i^2} = \frac{6+22i}{40} = \frac{3}{20} + \frac{11}{20}i$$

Argand Diagrams

There are two geometric representations of the complex number $z = x + iy$:

1. as the point $P(x, y)$ in the xy-plane

2. as the vector $\overrightarrow{OP}$ from the origin to P.

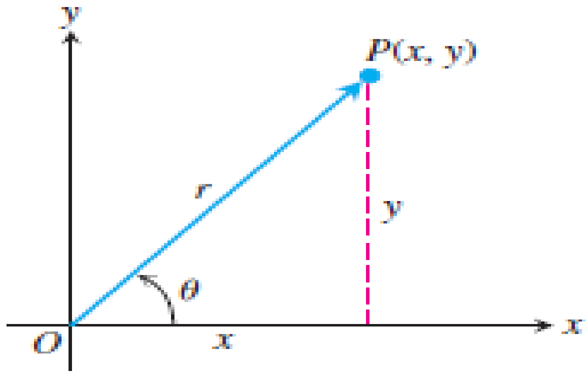

FIGURE A.4 This Argand diagram represents $z = x + iy$ both as a point $P(x, y)$ and as a vector $\overrightarrow{OP}$.

In each representation, the x-axis is called the **real axis** and the y-axis is the **imaginary axis**. Both representations are **Argand diagrams** for $x + iy$ (Figure A.4).

In terms of the polar coordinates of x and y, we have

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

and

$$z = x + iy = r(\cos\theta + i\sin\theta). \tag{10}$$

We define the **absolute value** of a complex number $x + iy$ to be the length r of a vector $\overrightarrow{OP}$ from the origin to $P(x, y)$. We denote the absolute value by vertical bars; thus,

$$|x + iy| = \sqrt{x^2 + y^2}.$$

If we always choose the polar coordinates r and θ so that r is nonnegative, then

$$r = |x + iy|.$$

The polar angle θ is called the **argument** of z and is written $\theta = \arg z$. Of course, any integer multiple of 2π may be added to θ to produce another appropriate angle.

The following equation gives a useful formula connecting a complex number z, its conjugate $\bar{z}$, and its absolute value $|z|$, namely,

$$z \cdot \bar{z} = |z|^2.$$

Euler’s Formula

The identity

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

called **Euler’s formula**, enables us to rewrite Equation (10) as

$$z = re^{i\theta}.$$

This formula, in turn, leads to the following rules for calculating products, quotients, powers, and roots of complex numbers. It also leads to Argand diagrams for $e^{i\theta}$. Since $\cos\theta + i\sin\theta$ is what we get from Equation (10) by taking $r = 1$, we can say that $e^{i\theta}$ is represented by a unit vector that makes an angle θ with the positive x-axis, as shown in Figure A.5.

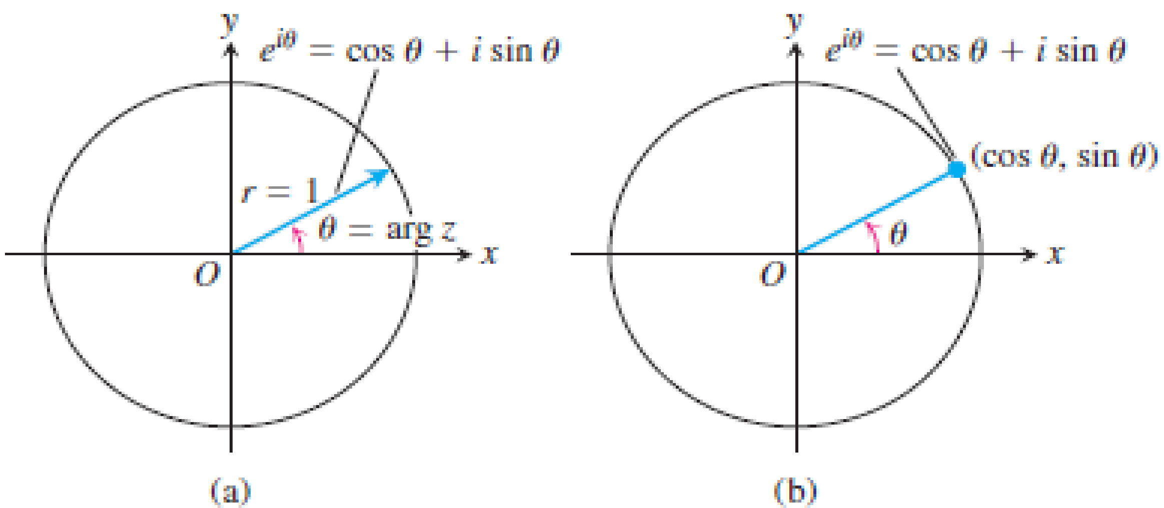

FIGURE A.5 Argand diagrams for $e^{i\theta} = \cos\theta + i\sin\theta$ (a) as a vector and (b) as a point.

Products

To multiply two complex numbers, we multiply their absolute values and add their angles. Let

$$z_1 = r_1 e^{i\theta_1}, \qquad z_2 = r_2 e^{i\theta_2}, \tag{11}$$

so that

$$|z_1| = r_1, \quad \arg z_1 = \theta_1; \quad |z_2| = r_2, \quad \arg z_2 = \theta_2.$$

Then

$$z_1 z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

and hence

$$\begin{aligned} |z_1 z_2| &= r_1 r_2 = |z_1| \cdot |z_2| \\ \arg(z_1 z_2) &= \theta_1 + \theta_2 = \arg z_1 + \arg z_2. \end{aligned} \tag{12}$$

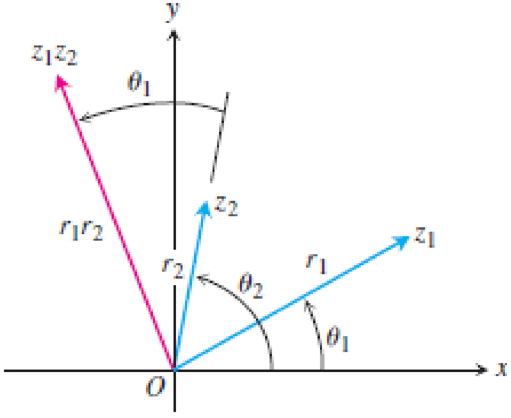

FIGURE A.6 When z_1 and z_2 are multiplied, $|z_1 z_2| = r_1 \cdot r_2$ and $\arg(z_1 z_2) = \theta_1 + \theta_2$.

Thus, the product of two complex numbers is represented by a vector whose length is the product of the lengths of the two factors and whose argument is the sum of their arguments (Figure A.6). In particular, from Equation (12) a vector may be rotated counterclockwise through an angle θ by multiplying it by $e^{i\theta}$. Multiplication by i rotates 90°, by -1 rotates 180°, by $-i$ rotates 270°, and so on.

EXAMPLE 2 Finding a Product of Complex Numbers

Let $z_1 = 1 + i, z_2 = \sqrt{3} - i$. We plot these complex numbers in an Argand diagram (Figure A.7) from which we read off the polar representations

$$z_1 = \sqrt{2}e^{i\pi/4}, \qquad z_2 = 2e^{-i\pi/6}.$$

Then

$$z_1 z_2 = 2\sqrt{2}\exp\left(\frac{i\pi}{4} - \frac{i\pi}{6}\right) = 2\sqrt{2}\exp\left(\frac{i\pi}{12}\right)$$

$$= 2\sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right) \approx 2.73 + 0.73i.$$

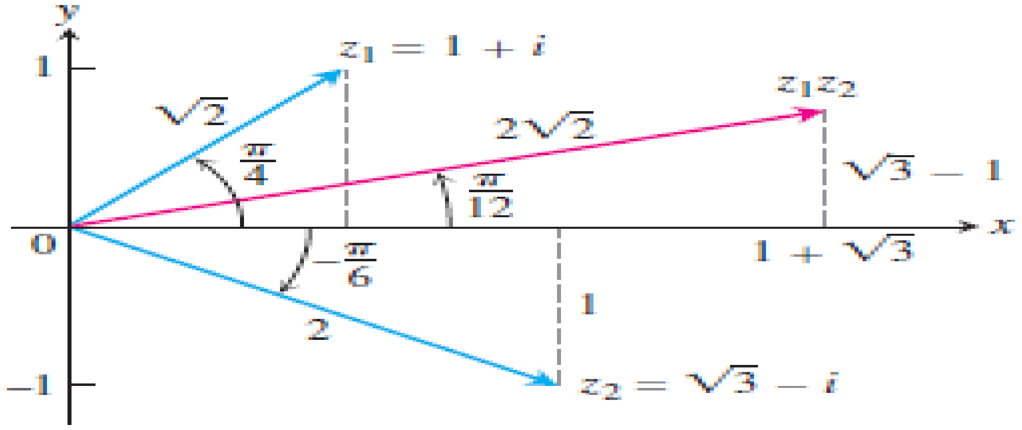

FIGURE A.7 To multiply two complex numbers, multiply their absolute values and add their arguments.

Quotients

Suppose $r_2 \neq 0$ in Equation (11). Then

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}.$$

Hence

$$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|} \quad \text{and} \quad \arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2.$$

That is, we divide lengths and subtract angles for the quotient of complex numbers.

EXAMPLE 3 Let $z_1 = 1 + i$ and $z_2 = \sqrt{3} - i$, as in Example 2. Then

$$\frac{1+i}{\sqrt{3}-i} = \frac{\sqrt{2}e^{i\pi/4}}{2e^{-i\pi/6}} = \frac{\sqrt{2}}{2} e^{5\pi i/12} \approx 0.707\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$$

$$\approx 0.183 + 0.683i.$$

Powers

If n is a positive integer, we may apply the product formulas in Equation (12) to find

$$z^n = z \cdot z \cdot \cdots \cdot z. \qquad n \text{ factors}$$

With $z = re^{i\theta}$, we obtain

$$z^n = (re^{i\theta})^n = r^n e^{i(\theta+\theta+\cdots+\theta)} \qquad n \text{ summands}$$

$$= r^n e^{in\theta}. \qquad (13)$$

The length $r = |z|$ is raised to the nth power and the angle $\theta = \arg z$ is multiplied by n.

If we take $r = 1$ in Equation (13), we obtain De Moivre's Theorem.

De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta. \qquad (14)$$

If we expand the left side of De Moivre's equation above by the Binomial Theorem and reduce it to the form $a + ib$, we obtain formulas for $\cos n\theta$ and $\sin n\theta$ as polynomials of degree n in $\cos\theta$ and $\sin\theta$.

EXAMPLE 4 If $n = 3$ in Equation (14), we have

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta.$$

The left side of this equation expands to

$$\cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta.$$

The real part of this must equal $\cos 3\theta$ and the imaginary part must equal $\sin 3\theta$. Therefore,

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta,$$

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta.$$

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Roots

If $z = re^{i\theta}$ is a complex number different from zero and n is a positive integer, then there are precisely n different complex numbers $w_0, w_1, \ldots, w_{n-1}$, that are nth roots of z. To see why, let $w = \rho e^{i\alpha}$ be an nth root of $z = re^{i\theta}$, so that

$$w^n = z$$

or

$$\rho^n e^{in\alpha} = re^{i\theta}.$$

Then

$$\rho = \sqrt[n]{r}$$

is the real, positive nth root of r. For the argument, although we cannot say that $n\alpha$ and θ must be equal, we can say that they may differ only by an integer multiple of 2π. That is,

$$n\alpha = \theta + 2k\pi, \qquad k = 0, \pm 1, \pm 2, \ldots.$$

Therefore,

$$\alpha = \frac{\theta}{n} + k\frac{2\pi}{n}.$$

Hence, all the nth roots of $z = re^{i\theta}$ are given by

$$\sqrt[n]{re^{i\theta}} = \sqrt[n]{r} \exp i\left(\frac{\theta}{n} + k\frac{2\pi}{n}\right), \qquad k = 0, \pm 1, \pm 2, \ldots. \qquad (15)$$

There might appear to be infinitely many different answers corresponding to the infinitely many possible values of k, but $k = n + m$ gives the same answer as $k = m$ in Equation (15). Thus, we need only take n consecutive values for k to obtain all the different nth roots of z. For convenience, we take

$$k = 0, 1, 2, \ldots, n - 1.$$

All the nth roots of $re^{i\theta}$ lie on a circle centered at the origin and having radius equal to the real, positive nth root of r. One of them has argument $\alpha = \theta/n$. The others are uniformly spaced around the circle, each being separated from its neighbors by an angle equal to $2\pi/n$. Figure A.8 illustrates the placement of the three cube roots, w_0, w_1, w_2, of the complex number $z = re^{i\theta}$.

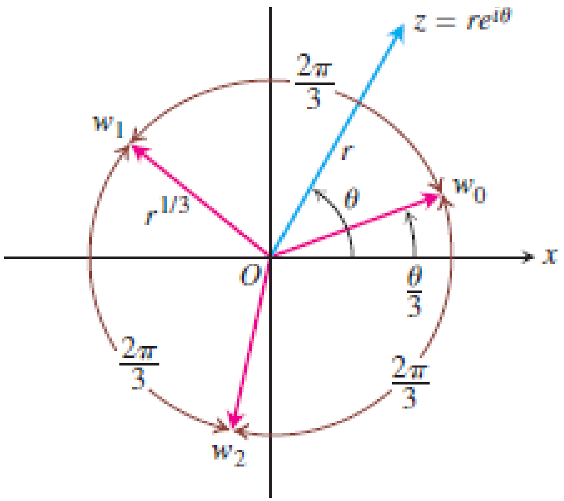

FIGURE A.8 The three cube roots of $z = re^{i\theta}$.

EXAMPLE 5 Finding Fourth Roots

Find the four fourth roots of -16.

Solution As our first step, we plot the number -16 in an Argand diagram (Figure A.9) and determine its polar representation $re^{i\theta}$. Here, $z = -16$, $r = +16$, and $\theta = \pi$. One of the fourth roots of $16e^{i\pi}$ is $2e^{i\pi/4}$. We obtain others by successive additions of $2\pi/4 = \pi/2$ to the argument of this first one. Hence,

$$\sqrt[4]{16 \exp i\pi} = 2 \exp i\left(\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right),$$

and the four roots are

$$w_0 = 2\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right] = \sqrt{2}(1+i)$$

$$w_1 = 2\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right] = \sqrt{2}(-1+i)$$

$$w_2 = 2\left[\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right] = \sqrt{2}(-1-i)$$

$$w_3 = 2\left[\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right] = \sqrt{2}(1-i).$$

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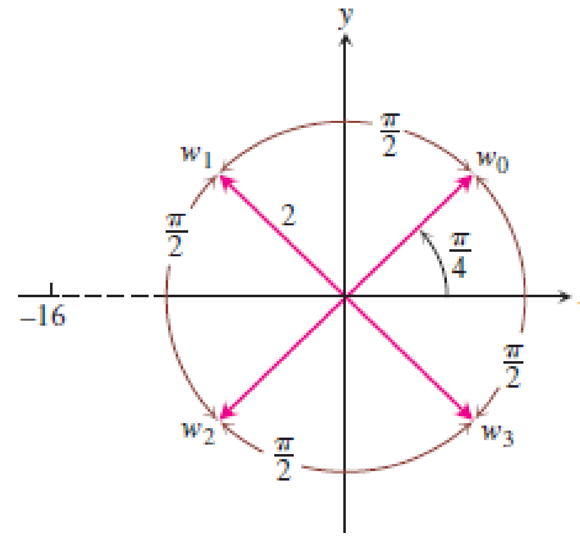

FIGURE A.9 The four fourth roots of -16.

EXERCISES 12.1

1. Solve the following equations for the real numbers, *x and y.*

a. $(3 + 4i)^2 - 2(x - iy) = x + iy$

b. $\left(\frac{1 + i}{1 - i}\right)^2 + \frac{1}{x + iy} = 1 + i$

c. $(3 - 2i)(x + iy) = 2(x - 2iy) + 2i - 1$

2. Express the complex numbers in Exercises 11–14 in the form $re^{i\theta}$, with $r \geq 0$ and $-\pi < \theta \leq \pi$. Draw an Argand diagram for each calculation.

11. $(1 + \sqrt{-3})^2$

12. $\frac{1 + i}{1 - i}$

13. $\frac{1 + i\sqrt{3}}{1 - i\sqrt{3}}$

14. $(2 + 3i)(1 - 2i)$

15. $\cos 4\theta$

16. $\sin 4\theta$

18. Find the two square roots of i.

19. Find the three cube roots of $-8i$.