MATHEMATICS II

SECOND SEMESTER

Lec. 10

TECHNIQUES OF INTEGRATION

Outlines

- Basic Integration Formulas
- Integration by Parts
- Tabular Integration
- Integration of Rational Functions by Partial Fractions
- Trigonometric Integrals

1. Products of Powers of Sines and Cosines
2. Integrals of Powers of tan *x and sec x*
3. Products of Sines and Cosines

10.1 Basic Integration Formulas.

In this section we present several algebraic or substitution methods to help us use this table 10.1.

TABLE 10.1 Basic integration formulas

1. $\int du = u + C$
2. $\int k\,du = ku + C$ (any number k)
3. $\int (du + dv) = \int du + \int dv$
4. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C$ $(n \neq -1)$
5. $\int \frac{du}{u} = \ln|u| + C$
6. $\int \sin u\,du = -\cos u + C$
7. $\int \cos u\,du = \sin u + C$
8. $\int \sec^2 u\,du = \tan u + C$
9. $\int \csc^2 u\,du = -\cot u + C$

10. $\int \sec u \tan u \, du = \sec u + C$

11. $\int \csc u \cot u \, du = -\csc u + C$

12. $\int \tan u \, du = -\ln |\cos u| + C$

$$= \ln |\sec u| + C$$

13. $\int \cot u \, du = \ln |\sin u| + C$

$$= -\ln |\csc u| + C$$

14. $\int e^u \, du = e^u + C$

15. $\int a^u \, du = \dfrac{a^u}{\ln a} + C \qquad (a > 0, a \neq 1)$

16. $\int \sinh u \, du = \cosh u + C$

17. $\int \cosh u \, du = \sinh u + C$

18. $\int \dfrac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\dfrac{u}{a}\right) + C$

19. $\int \dfrac{du}{a^2 + u^2} = \dfrac{1}{a} \tan^{-1}\left(\dfrac{u}{a}\right) + C$

20. $\int \dfrac{du}{u\sqrt{u^2 - a^2}} = \dfrac{1}{a} \sec^{-1}\left|\dfrac{u}{a}\right| + C$

21. $\int \dfrac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\dfrac{u}{a}\right) + C \qquad (a > 0)$

22. $\int \dfrac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\dfrac{u}{a}\right) + C \qquad (u > a > 0)$

TABLE 10.2 The secant and cosecant integrals

1. $$\int \sec u \, du = \ln |\sec u + \tan u| + C$$

2. $$\int \csc u \, du = -\ln |\csc u + \cot u| + C$$

10.2 Integration by Parts

In this section, we describe integration by parts and show how to apply it.

Product Rule in Integral Form

If f and g are differentiable functions of x, the Product Rule says

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

In terms of indefinite integrals, this equation becomes

$$\int \frac{d}{dx}[f(x)g(x)]\,dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx.$$

Rearranging the terms of this last equation, we get

$$\int f(x)g'(x)\,dx = \int \frac{d}{dx}[f(x)g(x)]\,dx - \int f'(x)g(x)\,dx$$

leading to the **integration by parts** formula

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx \qquad (1)$$

Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du \qquad (2)$$

EXAMPLE 1: Using Integration by Parts, Find $\int x \cos x \, dx$.

Solution We use the formula $\int u \, dv = uv - \int v \, du$ with

$$u = x, \qquad dv = \cos x \, dx,$$

$$du = dx, \qquad v = \sin x. \qquad \text{Simplest antiderivative of } \cos x$$

Then

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x + C.$$

EXAMPLE 2: Find $\int \ln x \, dx$.

Solution Since $\int \ln x \, dx$ can be written as $\int \ln x \cdot 1 \, dx$, we use the formula $\int u \, dv = uv - \int v \, du$ with

$u = \ln x$ Simplifies when differentiated $\quad dv = dx$ Easy to integrate

$du = \frac{1}{x} dx, \quad v = x.$ Simplest antiderivative

Then

$$\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - \int dx = x \ln x - x + C.$$

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EXAMPLE 3: Evaluate. $\displaystyle\int e^x \cos x \, dx$.

Solution Let $u = e^x$ and $dv = \cos x \, dx$. Then $du = e^x \, dx$, $v = \sin x$, and

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.$$

The second integral is like the first except that it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \qquad dv = \sin x \, dx, \qquad v = -\cos x, \qquad du = e^x \, dx.$$

Then

$$\int e^x \cos x \, dx = e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x \, dx)\right)$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx.$$

$$2\int e^x \cos x \, dx = e^x \sin x + e^x \cos x + C_1.$$

Dividing by 2 and renaming the constant of integration gives

$$\int e^x \cos x \, dx = \frac{e^x \sin x + e^x \cos x}{2} + C.$$

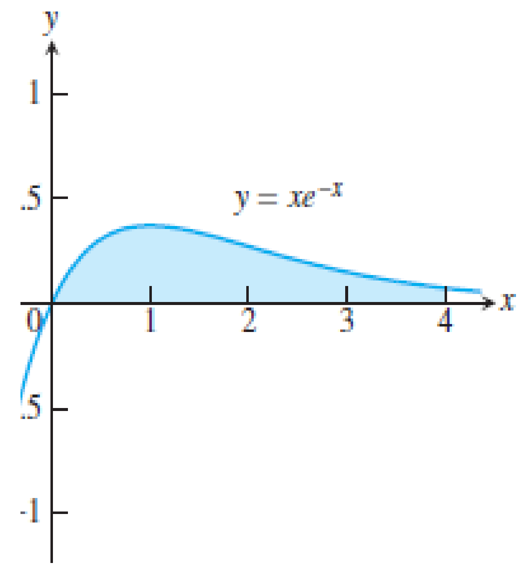
″Integration by Parts Formula for Definite Integrals

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx \qquad (3)$$

EXAMPLE 4: Find the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x = 0$ to $x = 4$.

Solution The region is shaded in Figure 8.1. Its area is

$$\int_0^4 xe^{-x}\,dx.$$



Let $u = x$, $dv = e^{-x}\,dx$, $v = -e^{-x}$, and $du = dx$. Then,

$$\int_0^4 xe^{-x}\,dx = -xe^{-x}\Big]_0^4 - \int_0^4 (-e^{-x})\,dx$$

$$= [-4e^{-4} - (0)] + \int_0^4 e^{-x}\,dx$$

$$= -4e^{-4} - e^{-x}\Big]_0^4$$

$$= -4e^{-4} - e^{-4} - (-e^0) = 1 - 5e^{-4} \approx 0.91.$$

SOLVED PROBLEMS

1. Find $\int x^3 e^{2x} dx$.

Let $u = x^3$, $dv = e^{2x}\,dx$. Then $du = 3x^2\,dx$, $v = \frac{1}{2}e^{2x}$, and

$$\int x^3 e^{2x} dx = \tfrac{1}{2}x^3 e^{2x} - \tfrac{3}{2}\int x^2 e^{2x} dx$$

For the resulting integral, let $u = x^2$ and $dv = e^{2x}\,dx$. Then $du = 2x\,dx$, $v = \frac{1}{2}e^{2x}$, and

$$\int x^3 e^{2x} dx = \tfrac{1}{2}x^3 e^{2x} - \tfrac{3}{2}\left(\tfrac{1}{2}x^2 e^{2x} - \int x e^{2x} dx\right) = \tfrac{1}{2}x^2 e^{2x} - \tfrac{3}{4}x^2 e^{2x} + \tfrac{3}{2}\int x e^{2x} dx$$

For the resulting integral, let $u = x$ and $dv = e^{2x}\,dx$. Then $du = dx$, $v = \frac{1}{2}e^{2x}$, and

$$\int x^3 e^{2x} dx = \tfrac{1}{2}x^3 e^{2x} - \tfrac{3}{4}x^2 e^{2x} + \tfrac{3}{2}\left(\tfrac{1}{2}x e^{2x} - \tfrac{1}{2}\int e^{2x} dx\right) = \tfrac{1}{2}x^3 e^{2x} - \tfrac{3}{4}x^2 e^{2x} + \tfrac{3}{4}x e^{2x} - \tfrac{3}{8}e^{2x} + C$$

2. Find $\int x^2 \ln x\, dx$.

Let $u = \ln x$, $dv = x^2\, dx$. Then $du = \frac{dx}{x}$, $v = \frac{x^3}{3}$, and

$$\int x^2 \ln x\, dx = \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\frac{dx}{x} = \frac{x^3}{3}\ln x - \tfrac{1}{3}\int x^2 dx = \frac{x^3}{3}\ln x - \tfrac{1}{9}x^3 + C$$

3. Find $\int \ln(x^2+2)\, dx$.

Let $u = \ln(x^2+2)$ and $dv = dx$.

$u = \ln(x^2+2)$	$dv = dx$
$du = \frac{2x}{x^2+2}dx$	$v = x$

So,

$$\begin{aligned}\int \ln(x^2+2)\, dx &= x\ln(x^2+2) - 2\int \frac{x^2}{x^2+2}\, dx \\ &= x\ln(x^2+2) - 2\int \left(1 - \frac{2}{x^2+2}\right)dx \\ &= x\ln(x^2+2) - 2x + \frac{4}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C \\ &= x(\ln(x^2+2) - 2) + 2\sqrt{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C\end{aligned}$$

4. Find $\int \sin^{-1} x\, dx$.

Let $u = \sin^{-1} x$, $dv = dx$.

$u = \sin^{-1} x$	$dv = dx$
$du = \dfrac{1}{\sqrt{1-x^2}}dx$	$v = x$

So,

$$\int \sin^{-1} x\, dx = x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}dx$$

$$= x\sin^{-1} x + \tfrac{1}{2}\int (1-x^2)^{-1/2}(-2x)dx$$

$$= x\sin^{-1} x + \tfrac{1}{2}(2(1-x^2)^{1/2}) + C$$

$$= x\sin^{-1} x + (1-x^2)^{1/2} + C = x\sin^{-1} x + \sqrt{1-x^2} + C$$

5. Find $\int \tan^{-1} x \, dx$.

Let $u = \tan^{-1}x$, $dv = dx$.

$u = \tan^{-1} x$	$dv = dx$
$du = \dfrac{1}{1+x^2}dx$	$v = x$

So,

$$\int \tan^{-1} x \, dx = x \tan^{-1} x - \int \frac{x}{1+x^2} dx = x \tan^{-1} x - \tfrac{1}{2}\int \frac{2x}{1+x^2} dx$$

$$= x \tan^{-1} x - \tfrac{1}{2}\ln(1+x^2) + C$$

10.3 Tabular Integration

tabular integration is illustrated in the following examples

EXAMPLE 4 : Using Tabular Integration , Evaluate $\int x^2 e^x \, dx.$

Solution With $f(x) = x^2$ and $g(x) = e^x$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^2	$(+)$	e^x
$2x$	$(-)$	e^x
2	$(+)$	e^x
0		e^x

We combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^2 e^x \, dx = x^2 e^x - 2x e^x + 2e^x + C.$$

EXAMPLE 5: Using Tabular Integration, Evaluate

$$\int x^3 \sin x \, dx.$$

Solution With $f(x) = x^3$ and $g(x) = \sin x$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^3	$(+)$	$\sin x$
$3x^2$	$(-)$	$-\cos x$
$6x$	$(+)$	$-\sin x$
6	$(-)$	$\cos x$
0		$\sin x$

Again we combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C. \quad \blacksquare$$

EXERCISES 10.1

1. Integration by Parts, Evaluate the integrals in Exercises 1–22.

1. $\int x \sin \frac{x}{2}\, dx$

2. $\int \theta \cos \pi\theta \, d\theta$

5. $\int_1^2 x \ln x \, dx$

6. $\int_1^e x^3 \ln x \, dx$

13. $\int (x^2 - 5x)e^x \, dx$

14. $\int (r^2 + r + 1)e^r \, dr$

21. $\int e^{\theta} \sin \theta \, d\theta$

22. $\int e^{-y} \cos y \, dy$

2. Evaluate the integrals in Exercises 25–30 by using a substitution prior to integration by parts.

25. $\int e^{\sqrt{3s+9}}\, ds$

26. $\int_0^1 x\sqrt{1-x}\, dx$

29. $\int \sin(\ln x)\, dx$

30. $\int z(\ln z)^2\, dz$

10.3 Integration of Rational Functions by Partial Fractions

This section shows how to express a rational function (a quotient of polynomials) as a sum of simpler fractions, called *partial fractions, which are easily integrated. For example,* the rational function (5*x - 3)/(x2 - 2x – 3)*can be rewritten as $\frac{5x-3}{x^2-2x-3} = \frac{2}{x+1} + \frac{3}{x-3}$,

The method for rewriting rational functions as a sum of simpler fractions is called **the method of partial fractions.** In the case of the above example, it consists of finding constants *A and B such that*

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}. \qquad (1)$$

To find *A and B, we first clear Equation (1) of fractions, obtaining*

$$5x - 3 = A(x-3) + B(x+1) = (A+B)x - 3A + B.$$

$$A + B = 5, \qquad -3A + B = -3.$$

Solving these equations simultaneously gives $A = 2$ and $B = 3$.

To integrate the rational function,

$$\int \frac{5x-3}{(x+1)(x-3)}\,dx = \int \frac{2}{x+1}\,dx + \int \frac{3}{x-3}\,dx$$

$$= 2\ln|x+1| + 3\ln|x-3| + C.$$

″General Description of the Method:

Success in writing a rational function *f(x) ⁄ g(x) as a sum of partial fractions depends on* two things:

- *The degree of f(x) must be less than the degree of g(x).*
- *We must know the factors of g(x).*

Here is how we find the partial fractions of a proper fraction *f(x)⁄g(x) when the factors of g are known.*

Method of Partial Fractions ($f(x)/g(x)$ Proper)

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x - r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m}.$$

Do this for each distinct linear factor of $g(x)$.

2. Let $x^2 + px + q$ be a quadratic factor of $g(x)$. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x + C_1}{x^2 + px + q} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_nx + C_n}{(x^2 + px + q)^n}.$$

Do this for each distinct quadratic factor of $g(x)$ that cannot be factored into linear factors with real coefficients.

3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.

4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

EXAMPLE 1: Evaluate , $\int \frac{x^2 + 4x + 1}{(x-1)(x+1)(x+3)}\,dx$ using partial fractions.

Solution : T $\frac{x^2 + 4x + 1}{(x-1)(x+1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}.$

To find the values of the undetermined coefficients *A, B, and C we* *cl*

$$x^2 + 4x + 1 = A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1)$$
$$= (A + B + C)x^2 + (4A + 2B)x + (3A - 3B - C).$$

So we equate coefficients of like powers of *x obtaining*

Coefficient of x^2: $A + B + C = 1$

Coefficient of x^1: $4A + 2B = 4$

Coefficient of x^0: $3A - 3B - C = 1$

the solution is $A = 3/4$, $B = 1/2$, and $C = -1/4$. Hence we have

$$\int \frac{x^2 + 4x + 1}{(x-1)(x+1)(x+3)}\,dx = \int \left[\frac{3}{4}\frac{1}{x-1} + \frac{1}{2}\frac{1}{x+1} - \frac{1}{4}\frac{1}{x+3}\right] dx$$

$$= \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3| + K,$$

where K is the arbitrary constant of integration (to avoid confusion with the undetermined coefficient we labeled as C). ■

EXAMPLE 2: Evaluate, $\int \frac{6x+7}{(x+2)^2}\,dx.$

Solution: First we express the integrand as a sum of partial fractions with undetermined coefficients.

$$\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

$$6x + 7 = A(x+2) + B \qquad \text{Multiply both sides by } (x+2)^2.$$

$$= Ax + (2A + B)$$

Equating coefficients of corresponding powers of *x gives*

$$A = 6 \quad \text{and} \quad 2A + B = 12 + B = 7, \quad \text{or} \quad A = 6 \quad \text{and} \quad B = -5.$$

Therefore,

$$\int \frac{6x+7}{(x+2)^2}\,dx = \int \left(\frac{6}{x+2} - \frac{5}{(x+2)^2}\right) dx$$

$$= 6\int \frac{dx}{x+2} - 5\int (x+2)^{-2}\,dx$$

$$= 6\ln|x+2| + 5(x+2)^{-1} + C$$

EXAMPLE 3: Integrating an Improper Fraction, Evaluate

Solution : First we divide the denominator into the numerator to get

a polynomial plus a proper fraction.

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\,dx.$$

$$\begin{array}{r l}
 & 2x \\
x^2 - 2x - 3 \overline{)} & 2x^3 - 4x^2 - x - 3 \\
 & \underline{2x^3 - 4x^2 - 6x} \\
 & \qquad\qquad 5x - 3
\end{array}$$

Then we write the improper fraction as a polynomial plus a proper fraction.

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}$$

We found the partial fraction decomposition of the fraction on the right in the opening example, so

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\,dx = \int 2x\,dx + \int \frac{5x - 3}{x^2 - 2x - 3}\,dx$$

$$= \int 2x\,dx + \int \frac{2}{x+1}\,dx + \int \frac{3}{x-3}\,dx$$

$$= x^2 + 2\ln|x+1| + 3\ln|x-3| + C.$$

EXAMPLE 4: Integrating with an Irreducible Quadratic Factor in the Denominator, Evaluate $\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx$ using partial fractions.

Solution: The denominator has an irreducible quadratic factor as well as a repeated linear factor, so we write

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}. \qquad (2)$$

Clearing the equation of fractions gives

$$\begin{aligned}-2x+4 &= (Ax+B)(x-1)^2 + C(x-1)(x^2+1) + D(x^2+1)\\ &= (A+C)x^3 + (-2A+B-C+D)x^2\\ &\quad + (A-2B+C)x + (B-C+D).\end{aligned}$$

Equating coefficients of like terms gives

$$\begin{aligned}&\text{Coefficients of } x^3: & 0 &= A + C\\ &\text{Coefficients of } x^2: & 0 &= -2A + B - C + D\\ &\text{Coefficients of } x^1: & -2 &= A - 2B + C\\ &\text{Coefficients of } x^0: & 4 &= B - C + D\end{aligned}$$

We solve these equations simultaneously to find the values of *A, B, C, and D:*

$$-4 = -2A, \qquad A = 2 \qquad \text{Subtract fourth equation from second.}$$

$$C = -A = -2 \qquad \text{From the first equation}$$

$$B = 1 \qquad A = 2 \text{ and } C = -2 \text{ in third equation.}$$

$$D = 4 - B + C = 1. \qquad \text{From the fourth equation}$$

We substitute these values into Equation (2), obtaining

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{2x + 1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2}.$$

Finally, using the expansion above we can integrate:

$$\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2}\,dx = \int \left(\frac{2x + 1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2}\right)dx$$

$$= \int \left(\frac{2x}{x^2 + 1} + \frac{1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2}\right)dx$$

$$= \ln(x^2 + 1) + \tan^{-1} x - 2\ln|x - 1| - \frac{1}{x - 1} + C.$$

EXAMPLE 5 A Repeated Irreducible Quadratic Factor

Evaluate

$$\int \frac{dx}{x(x^2+1)^2}.$$

Solution The form of the partial fraction decomposition is

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

Multiplying by $x(x^2+1)^2$, we have

$$\begin{aligned} 1 &= A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x \\ &= A(x^4+2x^2+1) + B(x^4+x^2) + C(x^3+x) + Dx^2 + Ex \\ &= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A \end{aligned}$$

If we equate coefficients, we get the system

$$A+B=0, \qquad C=0, \qquad 2A+B+D=0, \qquad C+E=0, \qquad A=1.$$

Solving this system gives $A=1$, $B=-1$, $C=0$, $D=-1$, and $E=0$. Thus,

$$\begin{aligned} \int \frac{dx}{x(x^2+1)^2} &= \int \left[\frac{1}{x} + \frac{-x}{x^2+1} + \frac{-x}{(x^2+1)^2}\right] dx \\ &= \int \frac{dx}{x} - \int \frac{x\,dx}{x^2+1} - \int \frac{x\,dx}{(x^2+1)^2} \\ &= \int \frac{dx}{x} - \frac{1}{2}\int \frac{du}{u} - \frac{1}{2}\int \frac{du}{u^2} \qquad u = x^2+1,\ du = 2x\,dx \\ &= \ln|x| - \frac{1}{2}\ln|u| + \frac{1}{2u} + K \\ &= \ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + K \end{aligned}$$

1. **Find** $\int \frac{dx}{x^2-4}$.

We factor the denominator into $(x-2)(x+2)$ and write $\frac{1}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2}$. Clearing of fractions yields

$$1 = A(x+2) + B(x-2) \qquad (1)$$

or

$$1 = (A+B)x + (2A-2B) \qquad (2)$$

Thus, $A + B = 0$ and $2A - 2B = 1$; $A = \frac{1}{4}$ and $B = -\frac{1}{4}$.

By either method, we have $\frac{1}{x^2-4} = \frac{\frac{1}{4}}{x-2} - \frac{\frac{1}{4}}{x+2}$. Then

$$\int \frac{dx}{x^2-4} = \frac{1}{4}\int \frac{dx}{x-2} - \frac{1}{4}\int \frac{dx}{x+2} = \frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + C = \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C$$

2. Find $\int \frac{(x+1)\,dx}{x^3+x^2-6x}$.

Factoring yields $x^3+x^2-6x = x(x-2)(x+3)$. Then $\frac{x+1}{x^3+x^2-6x} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+3}$ and

$$x+1 = A(x-2)(x+3) + Bx(x+3) + Cx(x-2) \qquad (1)$$

$$x+1 = (A+B+C)x^2 + (A+3B-2C)x - 6A \qquad (2)$$

General method: We solve simultaneously the system of equations

$$A+B+C=0 \qquad A+3B-2C=1 \qquad -6A=1$$

to obtain $A=-\frac{1}{6}$, $B=\frac{3}{10}$, and $C=-\frac{2}{15}$.

By either method,

$$\int \frac{(x+1)\,dx}{x^3+x^2-6x} = -\frac{1}{6}\int\frac{dx}{x} + \frac{3}{10}\int\frac{dx}{x-2} - \frac{2}{15}\int\frac{dx}{x+3}$$

$$= -\frac{1}{6}\ln|x| + \frac{3}{10}\ln|x-2| - \frac{2}{15}\ln|x+3| + C = \ln\frac{|x-2|^{3/10}}{|x|^{1/6}|x+3|^{2/15}} + C$$

3. Find $\int \frac{(3x+5)\,dx}{x^3-x^2-x+1}$.

$x^3-x^2-x+1=(x+1)(x-1)^2$. Hence, $\frac{3x+5}{x^3-x^2-x+1}=\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$ and

$$3x+5=A(x-1)^2+B(x+1)(x-1)+C(x+1)$$

For $x=-1$, $2=4A$ and $A=\frac{1}{2}$. For $x=1$, $8=2C$ and $C=4$. To determine the remaining constant, we use any other value of x, say $x=0$; for $x=0$, $5=A-B+C$ and $B=-\frac{1}{2}$. Thus,

$$\int \frac{3x+5}{x^3-x^2-x+1}\,dx=\frac{1}{2}\int\frac{dx}{x+1}-\frac{1}{2}\int\frac{dx}{x-1}+4\int\frac{dx}{(x-1)^2}$$

$$=\frac{1}{2}\ln|x+1|-\frac{1}{2}\ln|x-1|-\frac{4}{x-1}+C=-\frac{4}{x-1}+\frac{1}{2}\ln\left|\frac{x+1}{x-1}\right|+C$$

4. Find $\int \frac{x^3 + x^2 + x + 2}{x^4 + 3x^2 + 2}\, dx$.

$x^4 + 3x^2 + 2 = (x^2 + 1)(x^2 + 2)$. We write $\frac{x^3 + x^2 + x + 2}{x^4 + 3x^2 + 2} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{x^2 + 2}$ and obtain

$$x^3 + x^2 + x + 2 = (Ax + B)(x^2 + 2) + (Cx + D)(x^2 + 1)$$

$$= (A + C)x^3 + (B + D)x^2 + (2A + C)x + (2B + D)$$

Hence $A + C = 1$, $B + D = 1$, $2A + C = 1$, and $2B + D = 2$. Solving simultaneously yields $A = 0$, $B = 1$, $C = 1$, $D = 0$. Thus,

$$\int \frac{x^3 + x^2 + x + 2}{x^4 + 3x^2 + 2}\, dx = \int \frac{dx}{x^2 + 1} + \int \frac{x\, dx}{x^2 + 2} = \arctan x + \frac{1}{2} \ln (x^2 + 2) + C$$

5. **Find** $\int \frac{2x^2+3}{(x^2+1)^2}\,dx.$

We write $\frac{2x^2+3}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$. Then

$$2x^2+3 = (Ax+B)(x^2+1) + Cx + D = Ax^3 + Bx^2 + (A+C)x + (B+D)$$

from which $A=0$, $B=2$, $A+C=0$, $B+D=3$. Thus $A=0$, $B=2$, $C=0$, $D=1$ and

$$\int \frac{2x^2+3}{(x^2+1)^2}\,dx = \int \frac{2\,dx}{x^2+1} + \int \frac{dx}{(x^2+1)^2}$$

For the second integral on the right, let $x=\tan z$. Then

$$\int \frac{dx}{(x^2+1)^2} = \int \frac{\sec^2 z\,dz}{\sec^4 z} = \int \cos^2 z\,dz = \frac{1}{2}z + \frac{1}{4}\sin 2z + C$$

and $\int \frac{2x^2+3}{(x^2+1)^2}\,dx = 2\arctan x + \frac{1}{2}\arctan x + \frac{\frac{1}{2}x}{x^2+1} + C = \frac{5}{2}\arctan x + \frac{\frac{1}{2}x}{x^2+1} + C$

EXERCISES 10.2

1. Expand the quotients in Exercises 1–8 by partial fractions.

1. $\dfrac{5x-13}{(x-3)(x-2)}$ **2.** $\dfrac{5x-7}{x^2-3x+2}$ **7.** $\dfrac{t^2+8}{t^2-5t+6}$ **8.** $\dfrac{t^4+9}{t^4+9t^2}$

2. Nonrepeated Linear Factors, In Exercises 9–14, express the integrands as a sum of partial fractions and evaluate the integrals.

9. $\displaystyle\int \frac{dx}{1-x^2}$ **10.** $\displaystyle\int \frac{dx}{x^2+2x}$

3. Repeated Linear Factors, In Exercises, express the integrands as a sum of partial fractions and evaluate the integrals.

17. $\displaystyle\int_0^1 \frac{x^3\,dx}{x^2+2x+1}$ **18.** $\displaystyle\int_{-1}^0 \frac{x^3\,dx}{x^2-2x+1}$

10.4 Trigonometric Integrals

Trigonometric integrals involve algebraic combinations of the six basic trigonometric functions.

1. Products of Powers of Sines and Cosines

We begin with integrals of the form:

$$\int \sin^m x \cos^n x \, dx,$$

where m and n are nonnegative integers (positive or zero). We can divide the work into three cases.

Case 1 If m is odd, we write m as $2k+1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x. \qquad (1)$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x \, dx$ equal to $-d(\cos x)$.

Case 2 If m is even and n is odd in $\int \sin^m x \cos^n x\, dx$, we write n as $2k + 1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x.$$

We then combine the single $\cos x$ with dx and set $\cos x\, dx$ equal to $d(\sin x)$.

Case 3 If both m and n are even in $\int \sin^m x \cos^n x\, dx$, we substitute

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2} \tag{2}$$

EXAMPLE 1: *m is Odd,* Evaluate

$$\int \sin^3 x \cos^2 x\, dx.$$

Solution

$$\begin{aligned}
\int \sin^3 x \cos^2 x\, dx &= \int \sin^2 x \cos^2 x \sin x\, dx \\
&= \int (1 - \cos^2 x) \cos^2 x\, (-d(\cos x)) \\
&= \int (1 - u^2)(u^2)(-du) \qquad u = \cos x \\
&= \int (u^4 - u^2)\, du \\
&= \frac{u^5}{5} - \frac{u^3}{3} + C \\
&= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C.
\end{aligned}$$

EXAMPLE 2 *m* is Even and *n* is Odd

Evaluate

$$\int \cos^5 x \, dx.$$

Solution

$$\int \cos^5 x \, dx = \int \cos^4 x \cos x \, dx = \int (1 - \sin^2 x)^2 \, d(\sin x) \qquad m = 0$$

$$= \int (1 - u^2)^2 \, du \qquad u = \sin x$$

$$= \int (1 - 2u^2 + u^4) \, du$$

$$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C.$$

■

EXAMPLE 3 *m* and *n* are Both Even

Evaluate

$$\int \sin^2 x \cos^4 x \, dx .$$

Solution

$$\begin{aligned}
\int \sin^2 x \cos^4 x \, dx &= \int \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right)^2 dx \\
&= \frac{1}{8}\int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x)\, dx \\
&= \frac{1}{8}\int (1 + \cos 2x - \cos^2 2x - \cos^3 2x)\, dx \\
&= \frac{1}{8}\left[x + \frac{1}{2}\sin 2x - \int (\cos^2 2x + \cos^3 2x)\, dx\right].
\end{aligned}$$

For the term involving $\cos^2 2x$ we use

$$\begin{aligned}
\int \cos^2 2x \, dx &= \frac{1}{2}\int (1 + \cos 4x)\, dx \\
&= \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right).
\end{aligned}$$

Omitting the constant of integration until the final result

For the $\cos^3 2x$ term we have

$$\begin{aligned}
\int \cos^3 2x \, dx &= \int (1 - \sin^2 2x)\cos 2x \, dx \\
&= \frac{1}{2}\int (1 - u^2)\, du = \frac{1}{2}\left(\sin 2x - \frac{1}{3}\sin^3 2x\right).
\end{aligned}$$

$u = \sin 2x,$
$du = 2\cos 2x \, dx$

Again omitting C

Combining everything and simplifying we get

$$\int \sin^2 x \cos^4 x \, dx = \frac{1}{16}\left(x - \frac{1}{4}\sin 4x + \frac{1}{3}\sin^3 2x\right) + C.$$

■

Integrals of Powers of tan *x and sec x*

We know how to integrate the tangent and secant and their squares. To integrate higher powers we use the identities $\tan^2 x = \sec^2 x - 1$ and $\sec^2 x = \tan^2 x + 1$, and integrate by parts when necessary to reduce the higher powers to lower powers.

EXAMPLE 5 Evaluate

$$\int \tan^4 x \, dx.$$

Solution

$$\begin{aligned}
\int \tan^4 x \, dx &= \int \tan^2 x \cdot \tan^2 x \, dx = \int \tan^2 x \cdot (\sec^2 x - 1) \, dx \\
&= \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx \\
&= \int \tan^2 x \sec^2 x \, dx - \int (\sec^2 x - 1) \, dx \\
&= \int \tan^2 x \sec^2 x \, dx - \int \sec^2 x \, dx + \int dx.
\end{aligned}$$

In the first integral, we let

$$u = \tan x, \qquad du = \sec^2 x \, dx$$

and have

$$\int u^2 \, du = \frac{1}{3} u^3 + C_1.$$

The remaining integrals are standard forms, so

$$\int \tan^4 x \, dx = \frac{1}{3} \tan^3 x - \tan x + x + C.$$

2. Products of Sines and Cosines

The integrals

$$\int \sin mx \sin nx\, dx, \qquad \int \sin mx \cos nx\, dx, \qquad \text{and} \qquad \int \cos mx \cos nx\, dx$$

arise in many places where trigonometric functions are applied to problems in mathematics and science. We can evaluate these integrals through integration by parts, but two such integrations are required in each case. It is simpler to use the identities

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x], \tag{3}$$

$$\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x], \tag{4}$$

$$\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]. \tag{5}$$

EXAMPLE 7 Evaluate

$$\int \sin 3x \cos 5x \, dx.$$

Solution From Equation (4) with $m = 3$ and $n = 5$ we get

$$\begin{aligned}
\int \sin 3x \cos 5x \, dx &= \frac{1}{2} \int [\sin(-2x) + \sin 8x] \, dx \\
&= \frac{1}{2} \int (\sin 8x - \sin 2x) \, dx \\
&= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C.
\end{aligned}$$

EXERCISES 10.4

1. **Products of Powers of Sines and Cosines,** Evaluate the integrals in Exercises 1–13.

1. $\int_0^{\pi/2} \sin^5 x \, dx$

6. $\int_0^{\pi/2} 7\cos^7 t \, dt$

9. $\int_{-\pi/4}^{\pi/4} 16 \sin^2 x \cos^2 x \, dx$

13. $\int_0^{\pi/4} 8\cos^3 2\theta$

2. **Integrals with Square Roots,** Evaluate the integrals in Exercises 15–22.

15. $\int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}} \, dx$

16. $\int_0^{\pi} \sqrt{1-\cos 2x} \, dx$

21. $\int_0^{\pi/2} \theta\sqrt{1-\cos 2\theta} \, d\theta$

22. $\int_{-\pi}^{\pi} (1-\cos^2 t)^{3/2} \, dt$