MATHEMATICS II
SECOND SEMESTER

Lec. 09 Inverse Functions and Their Derivatives

Inverse Functions and Their Derivatives

A function that undoes, or inverts, the effect of a function *f* is called the *inverse* of *f*, Inverse functions play a key role in the development and properties of the logarithmic and exponential functions.

DEFINITION Inverse Function

Suppose that f is a one-to-one function on a domain D with range R. The **inverse function** f^{-1} is defined by

$$f^{-1}(a) = b \text{ if } f(b) = a.$$

The domain of f^{-1} is R and the range of f^{-1} is D.

$f^{-1}(x)$ does not mean $1/f(x)$.

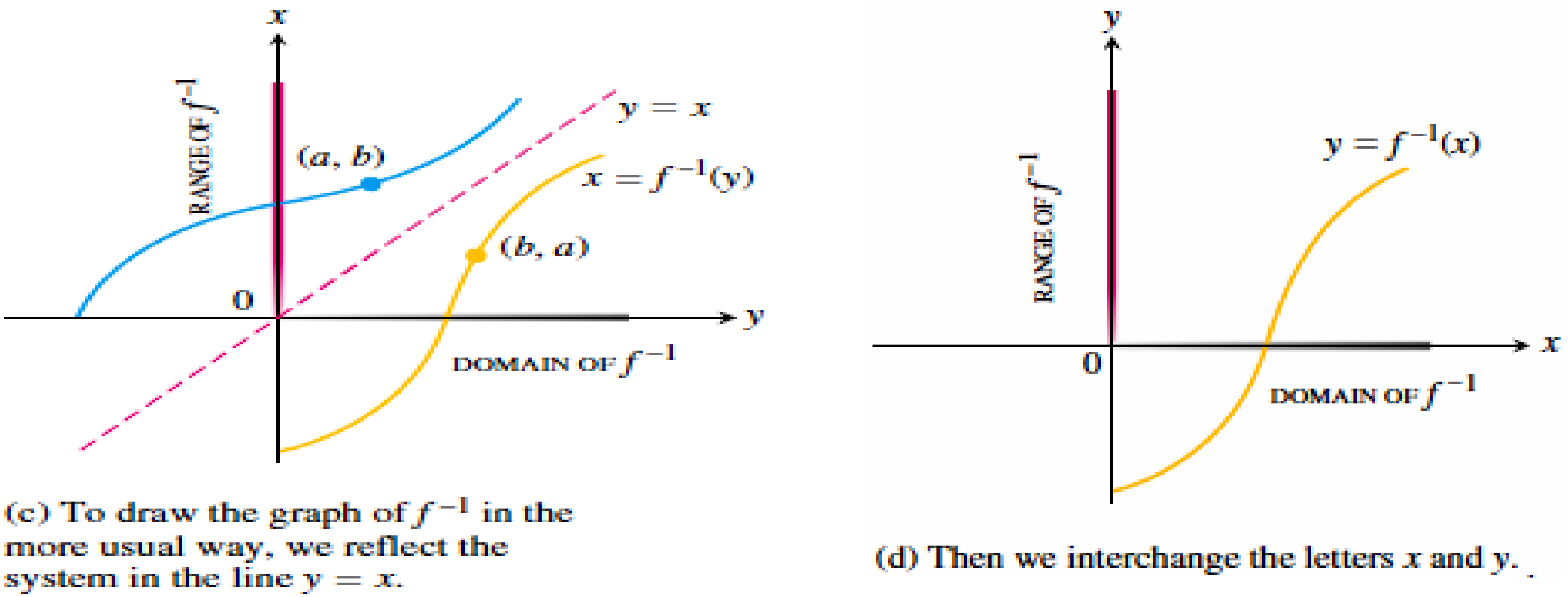


FIGURE 7.2 Determining the graph of $y = f^{-1}(x)$ from the graph of $y = f(x)$.

The process of passing from f to f^{-1} can be summarized as a two-step process.

1. Solve the equation $y = f(x)$ for x. This gives a formula $x = f^{-1}(y)$ where x is expressed as a function of y.
2. Interchange x and y, obtaining a formula $y = f^{-1}(x)$ where f^{-1} is expressed in the conventional format with x as the independent variable and y as the dependent variable.

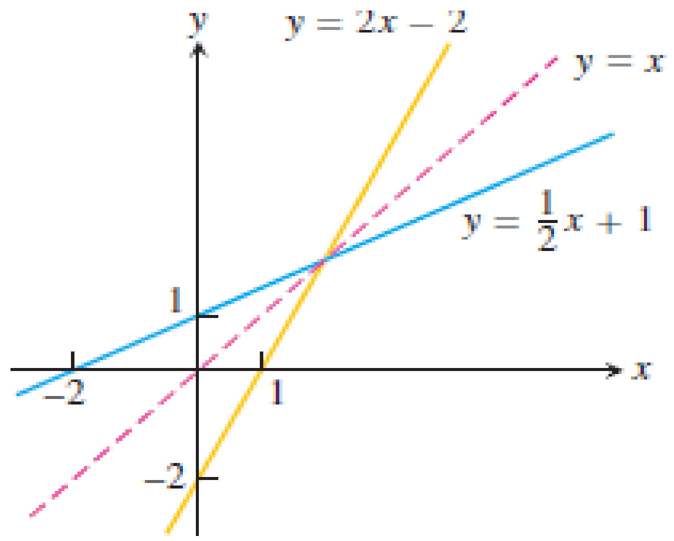
EXAMPLE 1: Finding an Inverse Function

Find the inverse of $y = \frac{1}{2}x + 1$, expressed as a function of x.

Solution

1. *Solve for x in terms of y:*

$$y = \frac{1}{2}x + 1$$
$$2y = x + 2$$
$$x = 2y - 2.$$

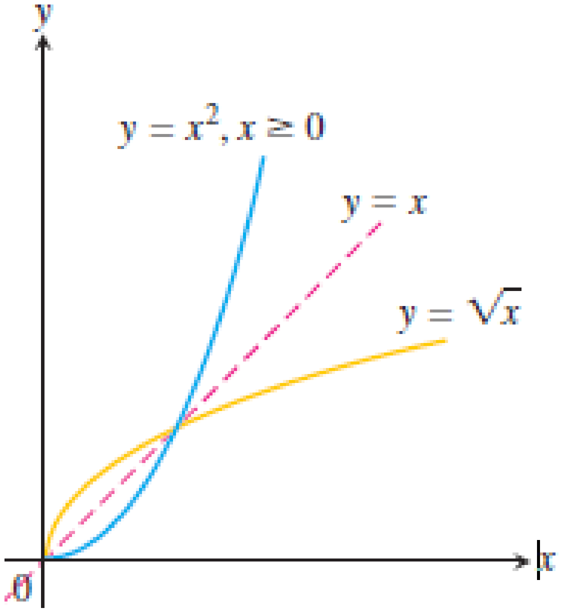


2. *Interchange x and y:* $y = 2x - 2.$

The inverse of the function $f(x) = (1/2)x + 1$ is the function $f^{-1}(x) = 2x - 2.$

EXAMPLE 2:

Find the inverse of the function $y = x^2, x \geq 0$, expressed as a function of x.



Solution We first solve for x in terms of y:

$$y = x^2$$

$$\sqrt{y} = \sqrt{x^2} = |x| = x \qquad |x| = x \text{ because } x \geq 0$$

We then interchange x and y, obtaining

$$y = \sqrt{x}.$$

The inverse of the function $y = x^2, x \geq 0$, is the function $y = \sqrt{x}$ (Figure 7.4).

˝Derivatives of Inverses of Differentiable Functions

If we calculate the derivatives of $f(x) = (1/2)x + 1$ and its inverse $f^{-1}(x) = 2x - 2$ from Example 1, we see that

$$\frac{d}{dx} f(x) = \frac{d}{dx}\left(\frac{1}{2}x + 1\right) = \frac{1}{2}$$

$$\frac{d}{dx} f^{-1}(x) = \frac{d}{dx}(2x - 2) = 2.$$

THEOREM 1 **The Derivative Rule for Inverses**

If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain. The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

or

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}} \tag{1}$$

EXAMPLE 3: Applying Theorem 1

The function $f(x) = x^2, x \geq 0$ and its inverse $f^{-1}(x) = \sqrt{x}$ have derivatives $f'(x) = 2x$ the derivative of $f^{-1}(x)$ is

Solution:

$$\begin{aligned}(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} \\ &= \frac{1}{2(f^{-1}(x))} \\ &= \frac{1}{2(\sqrt{x})}.\end{aligned}$$

EXAMPLE 4: Finding a Value of the Inverse Derivative

Let $f(x) = x^3 - 2$. Find the value of df^{-1}/dx at $x = 6 = f(2)$ without finding a formula for $f^{-1}(x)$.

Solution

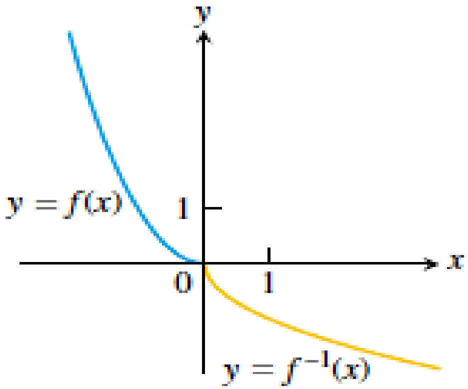
$$\left.\frac{df}{dx}\right|_{x=2} = \left.3x^2\right|_{x=2} = 12$$

$$\left.\frac{df^{-1}}{dx}\right|_{x=f(2)} = \frac{1}{\left.\frac{df}{dx}\right|_{x=2}} = \frac{1}{12} \qquad \text{Eq. (1)}$$

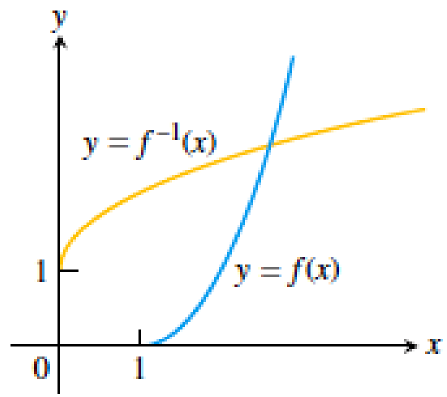
EXERCISES 9.1:

1. Each of Exercises, gives a formula for a function *y = f(x)* and shows the graphs of *f* and f^{-1}. *Find a formula for* f^{-1} *in each case.*

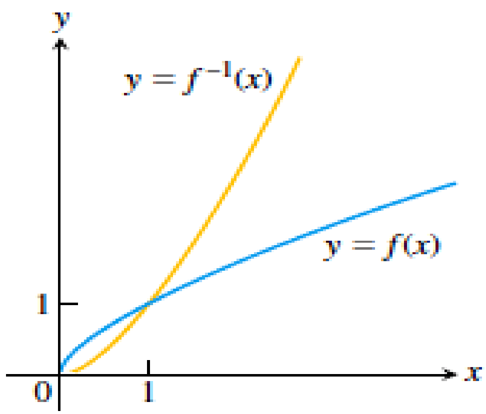
14. $f(x) = x^2, \quad x \le 0$



16. $f(x) = x^2 - 2x + 1, \quad x \ge 1$



18. $f(x) = x^{2/3}, \quad x \ge 0$



2. Formulas for Inverse Functions.

Each of Exercises 19–24 gives a formula for a function $y = f(x)$. In each case, find $f^{-1}(x)$ and identify the domain and range of f^{-1}. As a check, show that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

21. $f(x) = x^3 + 1$

22. $f(x) = (1/2)x - 7/2$

23. $f(x) = 1/x^2, \quad x > 0$

24. $f(x) = 1/x^3, \quad x \neq 0$

3.

Derivatives of Inverse Functions

In Exercises

a. Find $f^{-1}(x)$.

b. Graph f and f^{-1} together.

c. Evaluate df/dx at $x = a$ and df^{-1}/dx at $x = f(a)$ to show that at these points $df^{-1}/dx = 1/(df/dx)$.

25. $f(x) = 2x + 3, \quad a = -1$ **26.** $f(x) = (1/5)x + 7, \quad a = -1$

Natural Logarithms

In this section we use integral calculus to define the natural logarithm function, for which the number a is a particularly important value. The natural logarithm of a positive number x, written as ln x, is the value of an integral.

DEFINITION, The Natural Logarithm Function

$$\ln x = \int_1^x \frac{1}{x}\, dx,$$

DEFINITION The Number e

The number e is that number in the domain of the natural logarithm satisfying

$$\ln(e) = 1$$

TABLE 7.1 Typical 2-place values of $\ln x$

x	$\ln x$
0	undefined
0.05	−3.00
0.5	−0.69
1	0
2	0.69
3	1.10
4	1.39
10	2.30

The Derivative of ***y = ln x***

For every positive value of *x, we have* $\frac{d}{dx}\ln x = \frac{1}{x}$.

Therefore, the function $y = \ln x$ is a solution to the initial value problem $dy/dx = 1/x$,

Notice: If *u is a differentiable function of x whose values are positive, so that ln u is defined,* then applying the Chain Rule .

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

to the function $y = \ln u$ gives

$$\frac{d}{dx}\ln u = \frac{d}{du}\ln u \cdot \frac{du}{dx} = \frac{1}{u}\frac{du}{dx}.$$

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}, \qquad u > 0 \qquad (1)$$

EXAMPLE 1: Derivatives of Natural Logarithms

(a) $\frac{d}{dx}\ln 2x = \frac{1}{2x}\frac{d}{dx}(2x) = \frac{1}{2x}(2) = \frac{1}{x}$

(b) Equation (1) with $u = x^2 + 3$ gives

$$\frac{d}{dx}\ln(x^2+3) = \frac{1}{x^2+3}\cdot\frac{d}{dx}(x^2+3) = \frac{1}{x^2+3}\cdot 2x = \frac{2x}{x^2+3}.$$

Notice: The functio n *y = ln 2x* has the same derivative as the function *y = ln x.* This is true of *y = ln ax* for any positive number *a:*

$$\frac{d}{dx}\ln ax = \frac{1}{ax}\cdot\frac{d}{dx}(ax) = \frac{1}{ax}(a) = \frac{1}{x}. \qquad (2)$$

Since they have the same derivative, the functions *y = ln ax* and *y = ln x* differ by a constant.

Properties of Logarithms

For any numbers $a > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

1. *Product Rule:* $\ln ax = \ln a + \ln x$
2. *Quotient Rule:* $\ln \frac{a}{x} = \ln a - \ln x$
3. *Reciprocal Rule:* $\ln \frac{1}{x} = -\ln x$ Rule 2 with $a = 1$
4. *Power Rule:* $\ln x^r = r \ln x$ r rational

EXAMPLE 2: Interpreting the Properties of Logarithms

(a) $\ln 6 = \ln(2 \cdot 3) = \ln 2 + \ln 3$ Product

(b) $\ln 4 - \ln 5 = \ln \frac{4}{5} = \ln 0.8$ Quotient

(c) $\ln \frac{1}{8} = -\ln 8$ Reciprocal

$= -\ln 2^3 = -3 \ln 2$ Power

EXAMPLE 3: **Applying the Properties to Function Formulas**

(a) $\ln 4 + \ln \sin x = \ln(4 \sin x)$ Product

(b) $\ln \dfrac{x+1}{2x-3} = \ln(x+1) - \ln(2x-3)$ Quotient

(c) $\ln \sec x = \ln \dfrac{1}{\cos x} = -\ln \cos x$ Reciprocal

(d) $\ln \sqrt[3]{x+1} = \ln(x+1)^{1/3} = \dfrac{1}{3}\ln(x+1)$ Power

The Integral $\int (1/u)\, du$

If u is a differentiable function that is never zero,

$$\int \frac{1}{u}\, du = \ln |u| + C. \tag{5}$$

Equation (5) explains what to do when n equals -1. Equation (5) says integrals of a certain *form* lead to logarithms. If $u = f(x)$, then $du = f'(x)\, dx$ and

$$\int \frac{1}{u}\, du = \int \frac{f'(x)}{f(x)}\, dx.$$

So Equation (5) gives

$$\int \frac{f'(x)}{f(x)}\, dx = \ln |f(x)| + C$$

EXAMPLE 4 Applying Equation (5)

(a) $$\int_0^2 \frac{2x}{x^2-5}\,dx = \int_{-5}^{-1} \frac{du}{u} = \ln|u|\Big]_{-5}^{-1}$$

$u = x^2 - 5, \quad du = 2x\,dx,$
$u(0) = -5, \quad u(2) = -1$

$$= \ln|-1| - \ln|-5| = \ln 1 - \ln 5 = -\ln 5$$

(b) $$\int_{-\pi/2}^{\pi/2} \frac{4\cos\theta}{3+2\sin\theta}\,d\theta = \int_1^5 \frac{2}{u}\,du$$

$u = 3 + 2\sin\theta, \quad du = 2\cos\theta\,d\theta,$
$u(-\pi/2) = 1, \quad u(\pi/2) = 5$

$$= 2\ln|u|\Big]_1^5$$

$$= 2\ln|5| - 2\ln|1| = 2\ln 5$$

The Integrals of tan *x and cot x*

Equation (5) tells us at last how to integrate the tangent and cotangent functions.

$$\int \tan u \, du = -\ln|\cos u| + C = \ln|\sec u| + C$$

$$\int \cot u \, du = \ln|\sin u| + C = -\ln|\csc x| + C \qquad \text{..........(5)}$$

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = \int \frac{-du}{u} \qquad \begin{array}{l} u = \cos x > 0 \text{ on } (-\pi/2, \pi/2), \\ du = -\sin x \, dx \end{array}$$

$$= -\int \frac{du}{u} = -\ln|u| + C$$

$$= -\ln|\cos x| + C = \ln \frac{1}{|\cos x|} + C \qquad \text{Reciprocal Rule}$$

$$= \ln|\sec x| + C.$$

For the cotangent,

$$\int \cot x \, dx = \int \frac{\cos x \, dx}{\sin x} = \int \frac{du}{u} \qquad \begin{array}{l} u = \sin x, \\ du = \cos x \, dx \end{array}$$

$$= \ln|u| + C = \ln|\sin x| + C = -\ln|\csc x| + C.$$

EXAMPLE 5

$$\int_0^{\pi/6} \tan 2x \, dx = \int_0^{\pi/3} \tan u \cdot \frac{du}{2} = \frac{1}{2}\int_0^{\pi/3} \tan u \, du$$

$$= \frac{1}{2} \ln |\sec u| \Big]_0^{\pi/3} = \frac{1}{2}(\ln 2 - \ln 1) = \frac{1}{2} \ln 2$$

Substitute $u = 2x$,
$dx = du/2$,
$u(0) = 0$,
$u(\pi/6) = \pi/3$

EXERCISES 9.2

1. Using the Properties of Logarithms

a. Express the following logarithms in terms of ln 2 and ln 3.

a. $\ln 0.75$ b. $\ln(4/9)$ c. $\ln(1/2)$

d. $\ln \sqrt[3]{9}$ e. $\ln 3\sqrt{2}$ f. $\ln \sqrt{13.5}$

b. Use the properties of logarithms to simplify the expressions in Exercises 3 and 4.

3. a. $\ln \sin\theta - \ln\left(\dfrac{\sin\theta}{5}\right)$ b. $\ln(3x^2 - 9x) + \ln\left(\dfrac{1}{3x}\right)$

4. a. $\ln \sec\theta + \ln \cos\theta$ b. $\ln(8x + 4) - 2\ln 2$

c. $3\ln\sqrt[3]{t^2 - 1} - \ln(t + 1)$

2. In Exercises 8–22, find the derivative of *y with respect to x, t,or ɵ as* appropriate.

8. $y = \ln(t^{3/2})$

9. $y = \ln \frac{3}{x}$

12. $y = \ln(2\theta + 2)$

16. $y = t\sqrt{\ln t}$

17. $y = \frac{x^4}{4}\ln x - \frac{x^4}{16}$

18. $y = \frac{x^3}{3}\ln x - \frac{x^3}{9}$

14. $y = (\ln x)^3$

21. $y = \frac{\ln x}{1 + \ln x}$

22. $y = \frac{x \ln x}{1 + \ln x}$

31. $y = \ln(\sec(\ln \theta))$

32. $y = \ln\left(\frac{\sqrt{\sin\theta\cos\theta}}{1 + 2\ln\theta}\right)$

33. $y = \ln\left(\frac{(x^2+1)^5}{\sqrt{1-x}}\right)$

3. Evaluate the integrals in Exercises 37–54.

37. $\int_{-3}^{-2} \frac{dx}{x}$

38. $\int_{-1}^{0} \frac{3\,dx}{3x-2}$

54. $\int \frac{\sec x\,dx}{\sqrt{\ln(\sec x + \tan x)}}$

41. $\int_0^{\pi} \frac{\sin t}{2 - \cos t}\,dt$

42. $\int_0^{\pi/3} \frac{4\sin\theta}{1 - 4\cos\theta}\,d\theta$

46. $\int_2^{16} \frac{dx}{2x\sqrt{\ln x}}$

The Exponential Function

We study its properties and compute its derivative and integral.

Knowing its derivative, we prove the power rule to differentiate when *n is any* real number, rational or irrational.

The Function $y = e^x$

We can raise the number *e to a rational power r in the usual way:*

$$e^2 = e \cdot e, \qquad e^{-2} = \frac{1}{e^2}, \qquad e^{1/2} = \sqrt{e},$$

Typical values of e^x

x	e^x (rounded)
-1	0.37
0	1
1	2.72
2	7.39
10	22026
100	2.6881×10^{43}

DEFINITION The Natural Exponential Function

For every real number x, $e^x = \ln^{-1} x = \exp x$. (1)

Inverse Equations for e^x and $\ln x$

$$e^{\ln x} = x \qquad (\text{all } x > 0) \tag{2}$$

$$\ln(e^x) = x \qquad (\text{all } x) \tag{3}$$

Problem 1. Using a calculator, evaluate the following,

(a) $e^{2.731}$ (b) $e^{-3.162}$ (c) $\frac{5}{3}e^{5.253}$

(a) $e^{2.731} = 15.348227\ldots = \mathbf{15.348}$,

(b) $e^{-3.162} = 0.04234097\ldots = \mathbf{0.042341}$,

(c) $\frac{5}{3}e^{5.253} = \frac{5}{3}(191.138825\ldots) = \mathbf{318.56}$,

Problem 2. Use a calculator to determine the following,

(a) $3.72e^{0.18}$ (b) $53.2e^{-1.4}$ (c) $\frac{5}{122}e^{7}$

(a) $3.72e^{0.18} = (3.72)(1.197217\ldots) = \mathbf{4.454}$,
cant figures.

(b) $53.2e^{-1.4} = (53.2)(0.246596\ldots) = \mathbf{13.12}$,

(c) $\frac{5}{122}e^{7} = \frac{5}{122}(1096.6331\ldots) = \mathbf{44.94}$,

Graphs of exponential functions

Values of e^x and e^{-x} obtained from a calculator, correct to 2 decimal places, over a range $x = -3$ to $x = 3$, are shown in the following table.

x	−3.0	−2.5	−2.0	−1.5	−1.0	−0.5	0
e^x	0.05	0.08	0.14	0.22	0.37	0.61	1.00
e^{-x}	20.09	12.18	7.9	4.48	2.72	1.65	1.00

x	0.5	1.0	1.5	2.0	2.5	3.0
e^x	1.65	2.72	4.48	7.39	12.18	20.09
e^{-x}	0.61	0.37	0.22	0.14	0.08	0.05

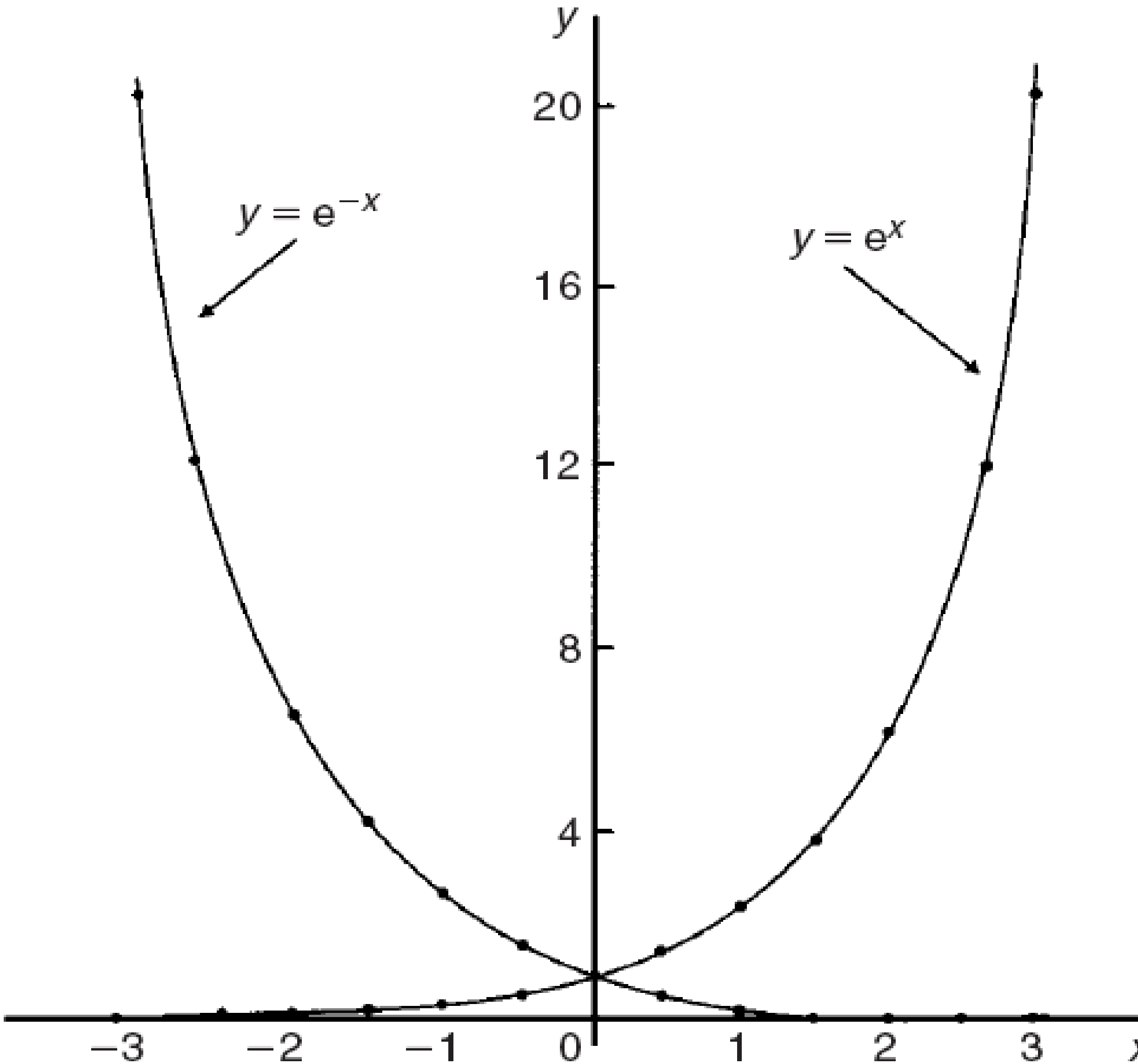


Figure shows graphs of $y = e^x$ and $y = e^{-x}$

The domain of $\ln x$ is $(0, \infty)$ and its range is $(-\infty, \infty)$. So the domain of e^x is $(-\infty, \infty)$ and its range is $(0, \infty)$.

EXAMPLE 1 Using the Inverse Equations

(a) $\ln e^2 = 2$

(b) $\ln e^{-1} = -1$

(c) $\ln \sqrt{e} = \frac{1}{2}$

(d) $\ln e^{\sin x} = \sin x$

(e) $e^{\ln 2} = 2$

(f) $e^{\ln (x^2+1)} = x^2 + 1$

(g) $e^{3 \ln 2} = e^{\ln 2^3} = e^{\ln 8} = 8$ One way

(h) $e^{3 \ln 2} = (e^{\ln 2})^3 = 2^3 = 8$ Another way

EXAMPLE 2 Solving for an Exponent

Find k if $e^{2k} = 10$.

Solution Take the natural logarithm of both sides:

$$\begin{aligned} e^{2k} &= 10 \\ \ln e^{2k} &= \ln 10 \\ 2k &= \ln 10 \qquad \text{Eq. (3)} \\ k &= \frac{1}{2}\ln 10. \end{aligned}$$

Problem 4. Given $20 = 60(1 - e^{-t/2})$ determine the value of t,

Rearranging $20 = 60(1 - e^{-t/2})$ gives:

$$\frac{20}{60} = 1 - e^{-1/2}$$

and

$$e^{-t/2} = 1 - \frac{20}{60} = \frac{2}{3}$$

Taking the reciprocal of both sides gives:

$$e^{t/2} = \frac{3}{2}$$

Taking Napierian logarithms of both sides gives:

$$\ln e^{t/2} = \ln \frac{3}{2}$$

i.e. $$\frac{t}{2} = \ln \frac{3}{2}$$

from which, $\boldsymbol{t} = 2 \ln \frac{3}{2} = \mathbf{0.881}$,

DEFINITION General Exponential Functions

For any numbers $a > 0$ and x, the exponential function with base a is

$$a^x = e^{x \ln a}.$$

EXAMPLE 3 Evaluating Exponential Functions

(a) $2^{\sqrt{3}} = e^{\sqrt{3} \ln 2} \approx e^{1.20} \approx 3.32$

(b) $2^{\pi} = e^{\pi \ln 2} \approx e^{2.18} \approx 8.8$

THEOREM 3 Laws of Exponents for e^x

For all numbers x, x_1, and x_2, the natural exponential e^x obeys the following laws:

1. $e^{x_1} \cdot e^{x_2} = e^{x_1 + x_2}$

2. $e^{-x} = \dfrac{1}{e^x}$

3. $\dfrac{e^{x_1}}{e^{x_2}} = e^{x_1 - x_2}$

4. $(e^{x_1})^{x_2} = e^{x_1 x_2} = (e^{x_2})^{x_1}$

EXAMPLE 4 Applying the Exponent Laws

(a) $e^{x+\ln 2} = e^x \cdot e^{\ln 2} = 2e^x$ Law 1

(b) $e^{-\ln x} = \dfrac{1}{e^{\ln x}} = \dfrac{1}{x}$ Law 2

(c) $\dfrac{e^{2x}}{e} = e^{2x-1}$ Law 3

(d) $(e^3)^x = e^{3x} = (e^x)^3$ Law 4

The Derivative and Integral of e^x

$$\frac{d}{dx}e^x = e^x \tag{5}$$

EXAMPLE 5 Differentiating an Exponential

$$\frac{d}{dx}(5e^x) = 5\frac{d}{dx}e^x$$
$$= 5e^x$$

The Chain Rule extends Equation (5) in the usual way to a more

If u is any differentiable function of x, then

$$\frac{d}{dx} e^{u} = e^{u} \frac{du}{dx}. \tag{6}$$

EXAMPLE 6 Applying the Chain Rule with Exponentials

(a) $\frac{d}{dx} e^{-x} = e^{-x} \frac{d}{dx}(-x) = e^{-x}(-1) = -e^{-x}$ Eq. (6) with $u = -x$

(b) $\frac{d}{dx} e^{\sin x} = e^{\sin x} \frac{d}{dx}(\sin x) = e^{\sin x} \cdot \cos x$ Eq. (6) with $u = \sin x$

The integral equivalent of Equation (6) is

$$\int e^{u}\, du = e^{u} + C.$$

EXAMPLE 7 Integrating Exponentials

(a) $$\int_0^{\ln 2} e^{3x}\,dx = \int_0^{\ln 8} e^u \cdot \frac{1}{3}\,du$$

$u = 3x,\quad \frac{1}{3}\,du = dx,\quad u(0) = 0,$
$u(\ln 2) = 3\ln 2 = \ln 2^3 = \ln 8$

$$= \frac{1}{3}\int_0^{\ln 8} e^u\,du$$

$$= \frac{1}{3}e^u\Big]_0^{\ln 8}$$

$$= \frac{1}{3}(8 - 1) = \frac{7}{3}$$

(b) $$\int_0^{\pi/2} e^{\sin x}\cos x\,dx = e^{\sin x}\Big]_0^{\pi/2}$$

Antiderivative from Example 6

$$= e^1 - e^0 = e - 1$$

EXERCISES 9.3

1. Solving Equations with Logarithmic or Exponential Terms

In Exercises 6–10, solve for *y in terms of t or x, as appropriate.*

6. $\ln y = -t + 5$

8. $\ln(1 - 2y) = t$

9. $\ln(y - 1) - \ln 2 = x + \ln x$

10. $\ln(y^2 - 1) - \ln(y + 1) = \ln(\sin x)$

In Exercises 13–16, solve for *t.*

13. a. $e^{-0.3t} = 27$

16. $e^{(x^2)}e^{(2x+1)} = e^t$

14. a. $e^{-0.01t} = 1000$

15. $e^{\sqrt{t}} = x^2$

2. In Exercises 23–32, find the derivative of *y with respect to x, t, or θ as* appropriate.

23. $y = (x^2 - 2x + 2)e^x$

24. $y = (9x^2 - 6x + 2)e^{3x}$

29. $y = \ln(3te^{-t})$

30. $y = \ln(2e^{-t}\sin t)$

25. $y = e^\theta(\sin\theta + \cos\theta)$

26. $y = \ln(3\theta e^{-\theta})$

31. $y = \ln\left(\frac{e^\theta}{1 + e^\theta}\right)$

32. $y = \ln\left(\frac{\sqrt{\theta}}{1 + \sqrt{\theta}}\right)$

In Exercises 37–40, find *dy/dx.*

37. $\ln y = e^{y} \sin x$

38. $\ln xy = e^{x+y}$

39. $e^{2x} = \sin(x + 3y)$

40. $\tan y = e^{x} + \ln x$

3. Evaluate the integrals in Exercises 41–62.

43. $\int_{\ln 2}^{\ln 3} e^{x}\, dx$

44. $\int_{\ln 2}^{0} e^{-x}\, dx$

47. $\int_{\ln 4}^{\ln 9} e^{x/2}\, dx$

48. $\int_{0}^{\ln 16} e^{x/4}\, dx$

55. $\int_{0}^{\pi/4} \left(1 + e^{\tan\theta}\right) \sec^2\theta\, d\theta$

56. $\int_{\pi/4}^{\pi/2} \left(1 + e^{\cot\theta}\right) \csc^2\theta\, d\theta$

59. $\int_{\ln(\pi/6)}^{\ln(\pi/2)} 2e^{v} \cos e^{v}\, dv$

60. $\int_{0}^{\sqrt{\ln \pi}} 2x\, e^{x^2} \cos\left(e^{x^2}\right) dx$

a^x and $\log_a x$

We have defined general exponential functions such as 2^x, 10^x, and π^x. In this section we compute their derivatives and integrals. We also define the general logarithmic functions such as $\log_2 x$, $\log_{10} x$, and $\log_\pi x$, and find their derivatives and integrals as well.

The Derivative of a^u

We start with the definition $a^x = e^{x \ln a}$:

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a} = e^{x \ln a} \cdot \frac{d}{dx}(x \ln a)$$
$$= a^x \ln a.$$

If $a > 0$, then

$$\frac{d}{dx} a^x = a^x \ln a.$$

With the Chain Rule, we get a more general form.

$$\frac{d}{dx} a^u = a^u \ln a \frac{du}{dx}. \qquad (1)$$

These equations show why e^x is the exponential function preferred in calculus. If $a = e$, then $\ln a = 1$ and the derivative of a^x simplifies to

$$\frac{d}{dx} e^x = e^x \ln e = e^x.$$

EXAMPLE 1 Differentiating General Exponential Functions

(a) $\dfrac{d}{dx} 3^x = 3^x \ln 3$

(b) $\dfrac{d}{dx} 3^{-x} = 3^{-x} (\ln 3) \dfrac{d}{dx}(-x) = -3^{-x} \ln 3$

(c) $\dfrac{d}{dx} 3^{\sin x} = 3^{\sin x} (\ln 3) \dfrac{d}{dx}(\sin x) = 3^{\sin x} (\ln 3) \cos x$

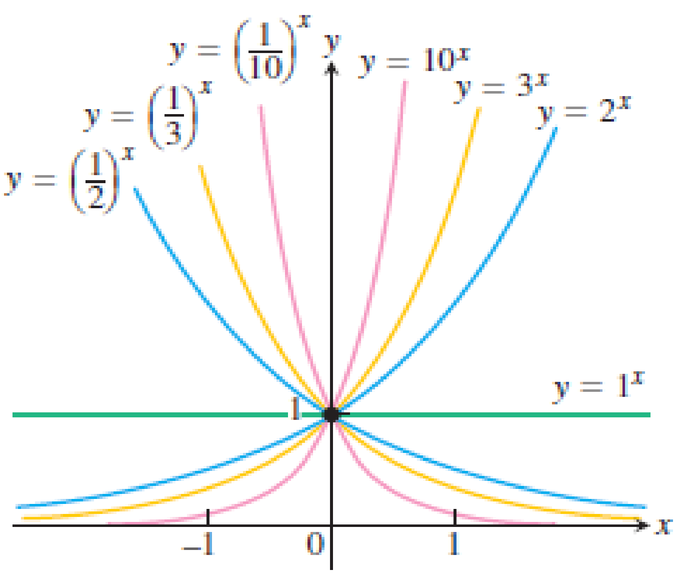


FIGURE 7.12 Exponential functions decrease if $0 < a < 1$ and increase if $a > 1$. As $x \to \infty$, we have $a^x \to 0$ if $0 < a < 1$ and $a^x \to \infty$ if $a > 1$. As $x \to -\infty$, we have $a^x \to \infty$ if $0 < a < 1$ and $a^x \to 0$ if $a > 1$.

The Integral of a^u

Writing the first integral in differential form gives

$$\int a^u \, du = \frac{a^u}{\ln a} + C. \qquad (2)$$

EXAMPLE 3 Integrating General Exponential Functions

(a) $\int 2^x \, dx = \dfrac{2^x}{\ln 2} + C$ Eq. (2) with $a = 2$, $u = x$

(b) $\int 2^{\sin x} \cos x \, dx$

$= \int 2^u \, du = \dfrac{2^u}{\ln 2} + C$ $u = \sin x$, $du = \cos x \, dx$, and Eq. (2)

$= \dfrac{2^{\sin x}}{\ln 2} + C$ u replaced by $\sin x$

Logarithms with Base a

DEFINITION $\log_a x$

For any positive number $a \neq 1$,

$$\log_a x \text{ is the inverse function of } a^x.$$

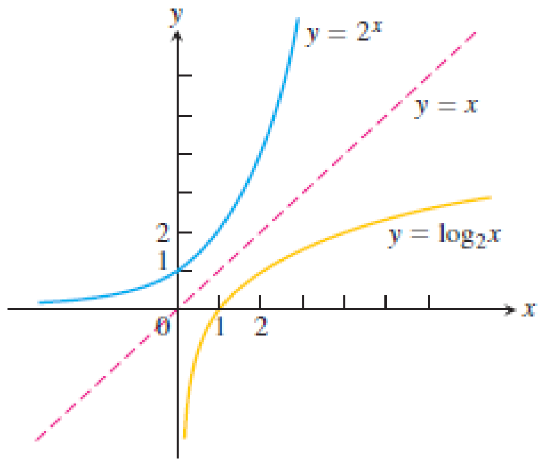


FIGURE 7.13 The graph of 2^x and its inverse, $\log_2 x$.

Inverse Equations for a^x and $\log_a x$

$$a^{\log_a x} = x \qquad (x > 0) \tag{3}$$

$$\log_a (a^x) = x \qquad (\text{all } x) \tag{4}$$

EXAMPLE 4 Applying the Inverse Equations

(a) $\log_2 (2^5) = 5$ **(b)** $\log_{10} (10^{-7}) = -7$

(c) $2^{\log_2 (3)} = 3$ **(d)** $10^{\log_{10} (4)} = 4$

Evaluation of $\log_a x$

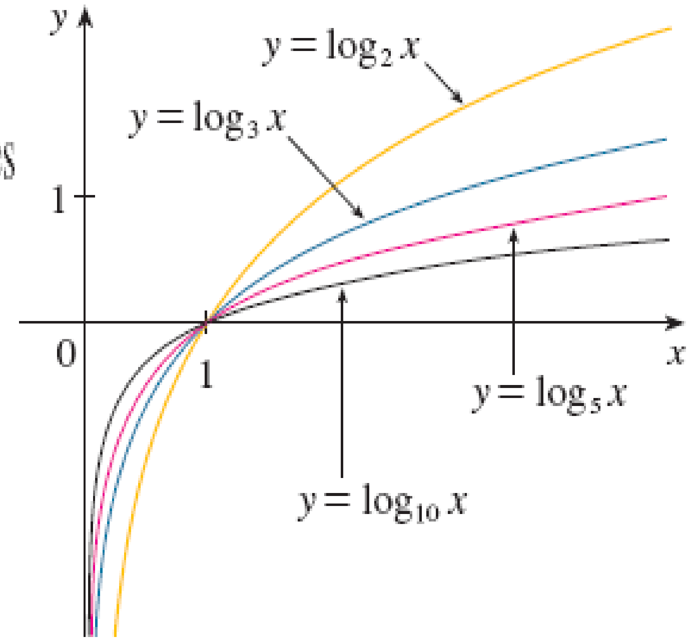
The evaluation of $\log_a x$ is simplified by the observation that $\log_a x$ is a numerical multiple of $\ln x$.

$$\log_a x = \frac{1}{\ln a} \cdot \ln x = \frac{\ln x}{\ln a} \qquad (5)$$

For example,

$$\log_{10} 2 = \frac{\ln 2}{\ln 10} \approx \frac{0.69315}{2.30259} \approx 0.30103$$

Figure shows the graphs of four logarithmic functions with various bases. In each case the domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the function increases slowly when $x > 1$.



The arithmetic rules satisfied by, given in Table 7.2

TABLE 7.2 Rules for base a logarithms

For any numbers $x > 0$ and $y > 0$,

1. *Product Rule:*
$$\log_a xy = \log_a x + \log_a y$$
2. *Quotient Rule:*
$$\log_a \frac{x}{y} = \log_a x - \log_a y$$
3. *Reciprocal Rule:*
$$\log_a \frac{1}{y} = -\log_a y$$
4. *Power Rule:*
$$\log_a x^y = y \log_a x$$

For example,

$$\ln xy = \ln x + \ln y$$ Rule 1 for natural logarithms …

$$\frac{\ln xy}{\ln a} = \frac{\ln x}{\ln a} + \frac{\ln y}{\ln a}$$ … divided by $\ln a$ …

$$\log_a xy = \log_a x + \log_a y.$$ … gives Rule 1 for base a logarithms.

Derivatives and Integrals Involving $\log_a x$

If u is a positive differentiable function of x, then

$$\frac{d}{dx}(\log_a u) = \frac{d}{dx}\left(\frac{\ln u}{\ln a}\right) = \frac{1}{\ln a}\frac{d}{dx}(\ln u) = \frac{1}{\ln a}\cdot\frac{1}{u}\frac{du}{dx}.$$

$$\frac{d}{dx}(\log_a u) = \frac{1}{\ln a}\cdot\frac{1}{u}\frac{du}{dx}$$

EXAMPLE 5

(a) $$\frac{d}{dx}\log_{10}(3x+1) = \frac{1}{\ln 10}\cdot\frac{1}{3x+1}\frac{d}{dx}(3x+1) = \frac{3}{(\ln 10)(3x+1)}$$

(b) $$\int \frac{\log_2 x}{x}\,dx = \frac{1}{\ln 2}\int \frac{\ln x}{x}\,dx \qquad \log_2 x = \frac{\ln x}{\ln 2}$$

$$= \frac{1}{\ln 2}\int u\,du \qquad u = \ln x, \quad du = \frac{1}{x}\,dx$$

$$= \frac{1}{\ln 2}\frac{u^2}{2} + C = \frac{1}{\ln 2}\frac{(\ln x)^2}{2} + C = \frac{(\ln x)^2}{2\ln 2} + C$$

EXERCISES 9.4

1. Algebraic Calculations With a^x and $\log_a x$

Simplify the expressions in Exercises 1–4.

1. a. $5^{\log_5 7}$ b. $8^{\log_8 \sqrt{2}}$ c. $1.3^{\log_{1.3} 75}$

   d. $\log_4 16$ e. $\log_3 \sqrt{3}$ f. $\log_4 \left(\frac{1}{4}\right)$

3. a. $2^{\log_4 x}$ b. $9^{\log_3 x}$ c. $\log_2\left(e^{(\ln 2)(\sin x)}\right)$

4. a. $25^{\log_5 (3x^2)}$ b. $\log_e (e^x)$ c. $\log_4\left(2^{e^x \sin x}\right)$

2. In Exercises, find the derivative of *y with respect to the given* independent variable.

11. $y = 2^x$

12. $y = 3^{-x}$

17. $y = (\cos\theta)^{\sqrt{2}}$

18. $y = (\ln\theta)^{\pi}$

29. $y = \log_3\left(\left(\frac{x+1}{x-1}\right)^{\ln 3}\right)$

30. $y = \log_5\sqrt{\left(\frac{7x}{3x+2}\right)^{\ln 5}}$

25. $y = \log_4 x + \log_4 x^2$

26. $y = \log_{25} e^x - \log_5\sqrt{x}$

3. Evaluate the integrals in Exercises.

47. $\int 5^x \, dx$

52. $\int_1^4 \frac{2^{\sqrt{x}}}{\sqrt{x}} \, dx$

53. $\int_0^{\pi/2} 7^{\cos t} \sin t \, dt$

67. $\int_0^9 \frac{2\log_{10}(x+1)}{x+1} \, dx$

Inverse Trigonometric Functions

- Inverse trigonometric functions arise when we want to calculate angles from side measurements in triangles. This section shows how these functions are defined,graphed, and evaluated, how their derivatives are computed.

$$y = \sin^{-1} x \quad \text{or} \quad y = \arcsin x$$

$$y = \cos^{-1} x \quad \text{or} \quad y = \arccos x$$

$$y = \tan^{-1} x \quad \text{or} \quad y = \arctan x$$

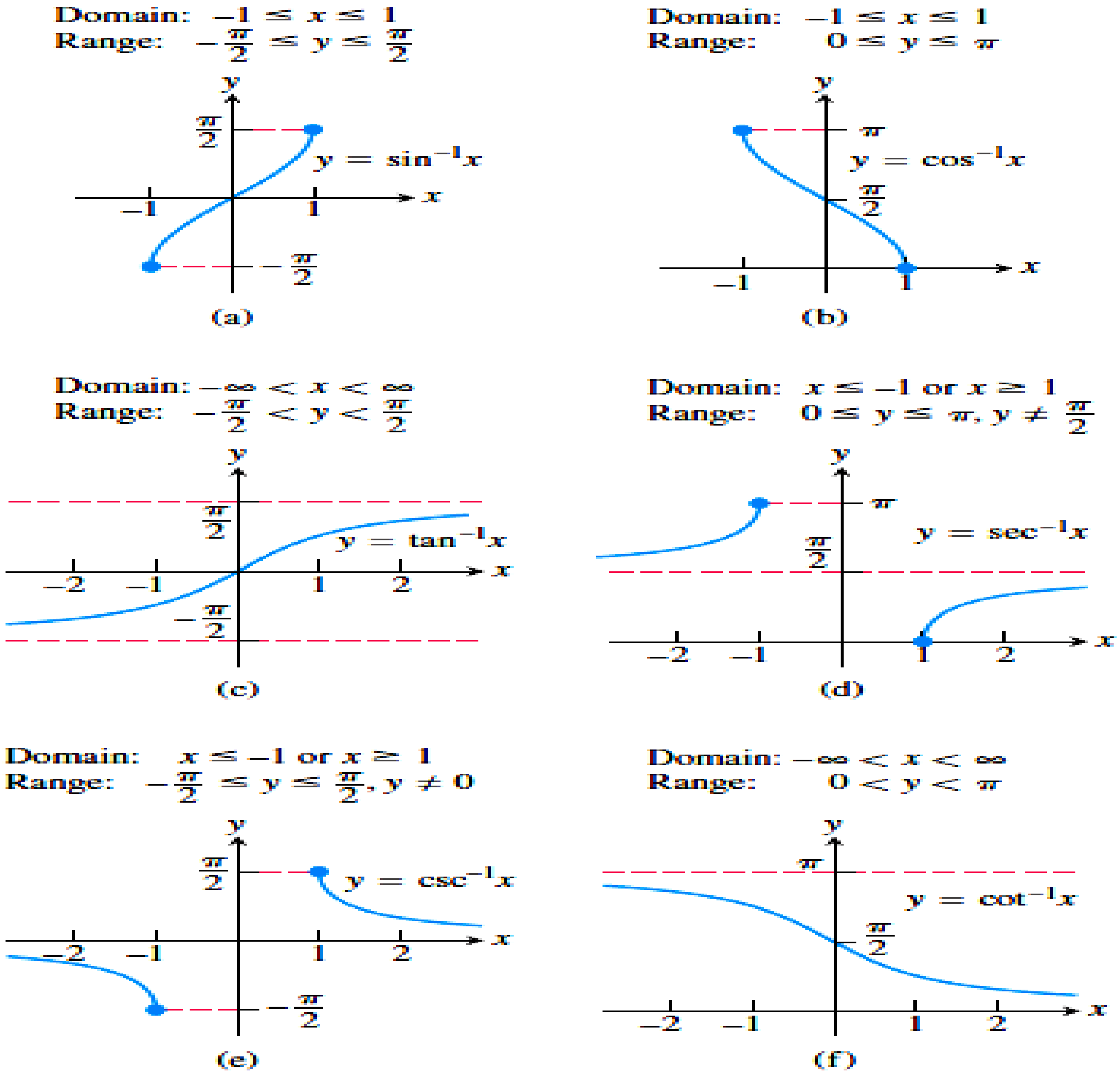
$$y = \cot^{-1} x \quad \text{or} \quad y = \operatorname{arccot} x$$

$$y = \sec^{-1} x \quad \text{or} \quad y = \operatorname{arcsec} x$$

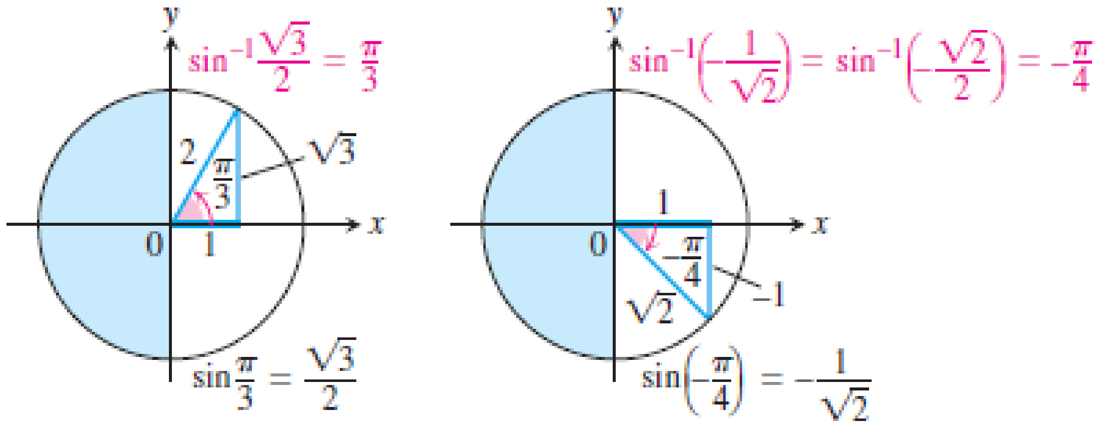
$$y = \csc^{-1} x \quad \text{or} \quad y = \operatorname{arccsc} x$$

These equations are read "y equals the arcsine of x" or "y equals arcsin x" and so on.

The graphs of the six inverse trigonometric functions are shown in Figure



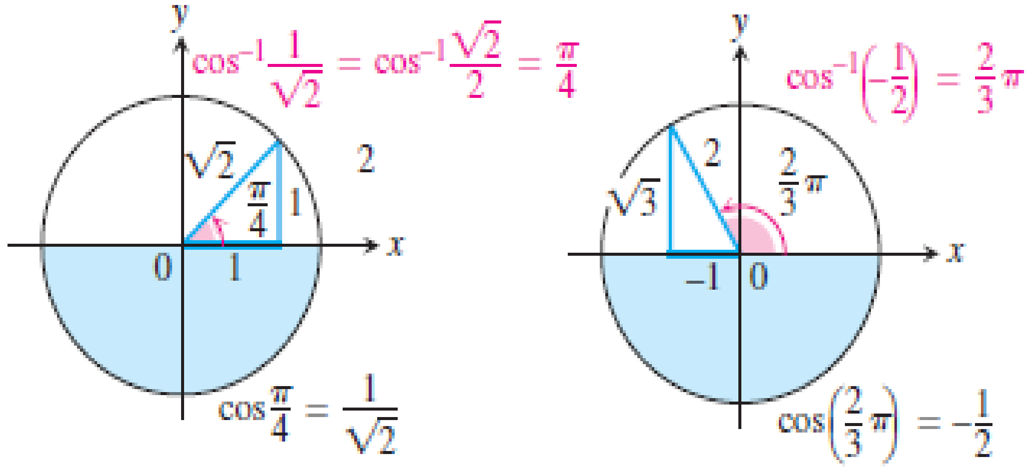
EXAMPLE 1 Common Values of $\sin^{-1} x$



The angles come from the first and fourth quadrants because the range of $\sin^{-1} x$ is $[-\pi/2, \pi/2]$. ■

x	$\sin^{-1} x$
$\sqrt{3}/2$	$\pi/3$
$\sqrt{2}/2$	$\pi/4$
$1/2$	$\pi/6$
$-1/2$	$-\pi/6$
$-\sqrt{2}/2$	$-\pi/4$
$-\sqrt{3}/2$	$-\pi/3$

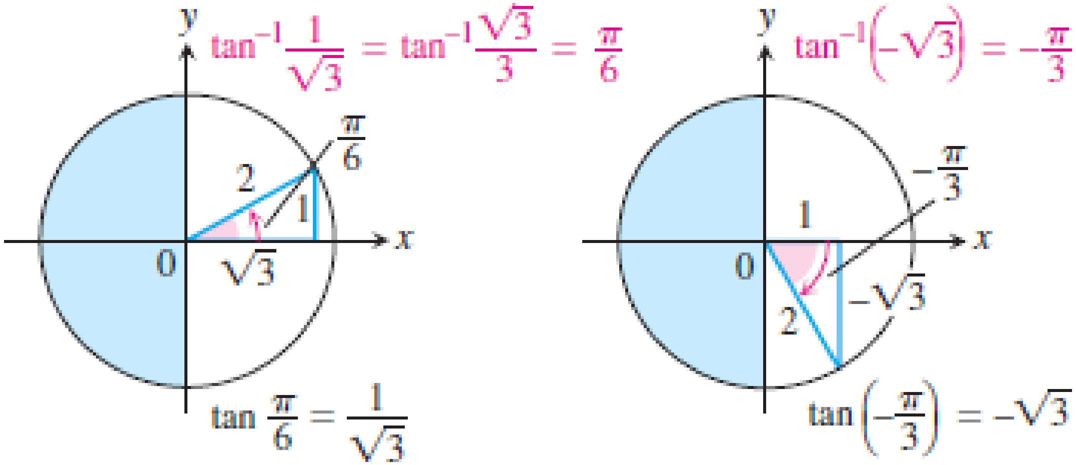
EXAMPLE 2 Common Values of $\cos^{-1} x$



The angles come from the first and second quadrants because the range of $\cos^{-1} x$ is $[0, \pi]$. ■

x	$\cos^{-1} x$
$\sqrt{3}/2$	$\pi/6$
$\sqrt{2}/2$	$\pi/4$
$1/2$	$\pi/3$
$-1/2$	$2\pi/3$
$-\sqrt{2}/2$	$3\pi/4$
$-\sqrt{3}/2$	$5\pi/6$

EXAMPLE 3 Common Values of $\tan^{-1} x$



The angles come from the first and fourth quadrants because the range of $\tan^{-1} x$ is $(-\pi/2, \pi/2)$. ■

x	$\tan^{-1} x$
$\sqrt{3}$	$\pi/3$
1	$\pi/4$
$\sqrt{3}/3$	$\pi/6$
$-\sqrt{3}/3$	$-\pi/6$
-1	$-\pi/4$
$-\sqrt{3}$	$-\pi/3$

EXAMPLE 4 Find $\cos \alpha$, $\tan \alpha$, $\sec \alpha$, $\csc \alpha$, and $\cot \alpha$ if

$$\alpha = \sin^{-1} \frac{2}{3}.$$

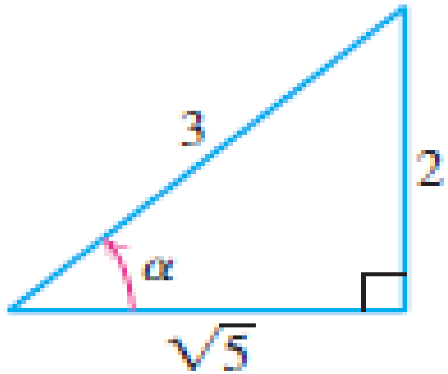
Solution This equation says that $\sin \alpha = 2/3$. We picture α as an angle in a right triangle with opposite side 2 and hypotenuse 3 (Figure 7.27). The length of the remaining side is

$$\sqrt{(3)^2 - (2)^2} = \sqrt{9 - 4} = \sqrt{5}. \qquad \text{Pythagorean theorem}$$

We add this information to the figure and then read the values we want from the completed triangle:

$$\cos \alpha = \frac{\sqrt{5}}{3}, \quad \tan \alpha = \frac{2}{\sqrt{5}}, \quad \sec \alpha = \frac{3}{\sqrt{5}}, \quad \csc \alpha = \frac{3}{2}, \quad \cot \alpha = \frac{\sqrt{5}}{2}.$$

■



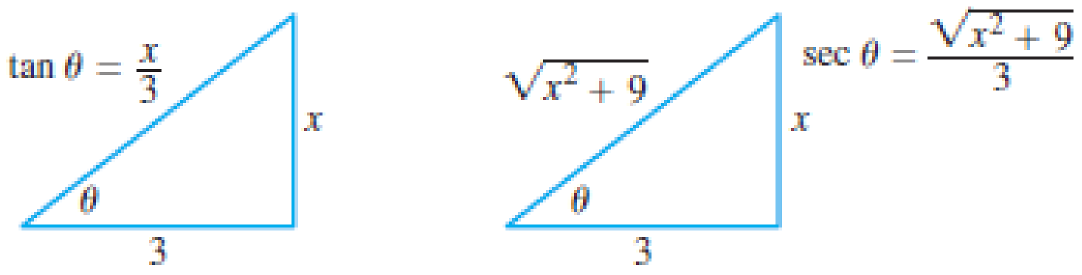
EXAMPLE 5 Find $\sec\left(\tan^{-1}\frac{x}{3}\right)$.

Solution We let $\theta = \tan^{-1}(x/3)$ (to give the angle a name) and picture θ in a right triangle with

$$\tan\theta = \text{opposite/adjacent} = x/3.$$

The length of the triangle's hypotenuse is

$$\sqrt{x^2 + 3^2} = \sqrt{x^2 + 9}.$$



Thus,

$$\sec\left(\tan^{-1}\frac{x}{3}\right) = \sec\theta$$

$$= \frac{\sqrt{x^2 + 9}}{3}. \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

■

The Derivative of $y = \sin^{-1} u$

If u is a differentiable function of x with $|u| < 1$, we apply the Chain Rule to get

$$\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1 - u^2}}\frac{du}{dx}, \qquad |u| < 1.$$

EXAMPLE 7 Applying the Derivative Formula

$$\frac{d}{dx}(\sin^{-1} x^2) = \frac{1}{\sqrt{1 - (x^2)^2}} \cdot \frac{d}{dx}(x^2) = \frac{2x}{\sqrt{1 - x^4}}$$

The Derivative of $y = \tan^{-1} u$

The derivative is defined for all real numbers. If u is a differentiable function of x, we get the Chain Rule form:

$$\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1 + u^2}\frac{du}{dx}.$$

The Derivative of $y = \sec^{-1} u$

If u is a differentiable function of x with $|u| > 1$, we have the formula

$$\frac{d}{dx}(\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2 - 1}}\frac{du}{dx}, \qquad |u| > 1.$$

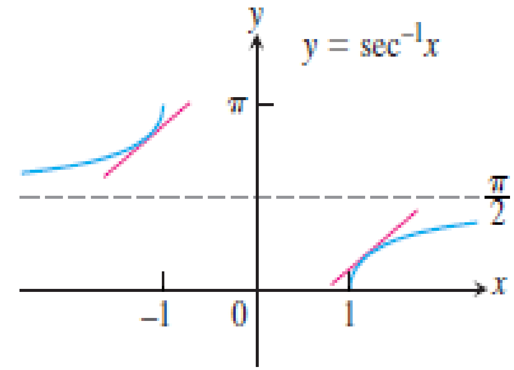


FIGURE 7.30 The slope of the curve $y = \sec^{-1} x$ is positive for both $x < -1$ and $x > 1$.

EXAMPLE 9 Using the Formula

$$\begin{aligned}\frac{d}{dx}\sec^{-1}(5x^4) &= \frac{1}{|5x^4|\sqrt{(5x^4)^2 - 1}}\frac{d}{dx}(5x^4) \\ &= \frac{1}{5x^4\sqrt{25x^8 - 1}}(20x^3) \qquad 5x^4 > 0 \\ &= \frac{4}{x\sqrt{25x^8 - 1}}\end{aligned}$$

Derivatives of the Other Three

Inverse Function–Inverse Cofunction Identities

$$\cos^{-1} x = \pi/2 - \sin^{-1} x$$
$$\cot^{-1} x = \pi/2 - \tan^{-1} x$$
$$\csc^{-1} x = \pi/2 - \sec^{-1} x$$

For example, the derivative of $\cos^{-1} x$ is calculated as follows:

$$\frac{d}{dx}(\cos^{-1} x) = \frac{d}{dx}\left(\frac{\pi}{2} - \sin^{-1} x\right) \qquad \text{Identity}$$

$$= -\frac{d}{dx}(\sin^{-1} x)$$

$$= -\frac{1}{\sqrt{1 - x^2}} \qquad \text{Derivative of arcsine}$$

EXAMPLE 10:

Find an equation for the line tangent to the graph of $y = \cot^{-1} x$ at $x = -1$.

Solution First we note that

$$\cot^{-1}(-1) = \pi/2 - \tan^{-1}(-1) = \pi/2 - (-\pi/4) = 3\pi/4.$$

The slope of the tangent line is

$$\left.\frac{dy}{dx}\right|_{x=-1} = -\left.\frac{1}{1+x^2}\right|_{x=-1} = -\frac{1}{1+(-1)^2} = -\frac{1}{2},$$

so the tangent line has equation $y - 3\pi/4 = (-1/2)(x + 1)$.

TABLE 7.3 Derivatives of the inverse trigonometric functions

1. $$\frac{d(\sin^{-1} u)}{dx} = \frac{du/dx}{\sqrt{1 - u^2}}, \quad |u| < 1$$

2. $$\frac{d(\cos^{-1} u)}{dx} = -\frac{du/dx}{\sqrt{1 - u^2}}, \quad |u| < 1$$

3. $$\frac{d(\tan^{-1} u)}{dx} = \frac{du/dx}{1 + u^2}$$

4. $$\frac{d(\cot^{-1} u)}{dx} = -\frac{du/dx}{1 + u^2}$$

5. $$\frac{d(\sec^{-1} u)}{dx} = \frac{du/dx}{|u|\sqrt{u^2 - 1}}, \quad |u| > 1$$

6. $$\frac{d(\csc^{-1} u)}{dx} = \frac{-du/dx}{|u|\sqrt{u^2 - 1}}, \quad |u| > 1$$

Integration Formulas

The derivative formulas in Table 7.3 yield three useful integration formulas in Table 7.4.

TABLE 7.4 Integrals evaluated with inverse trigonometric functions

The following formulas hold for any constant $a \neq 0$.

1. $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$ (Valid for $u^2 < a^2$)

2. $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$ (Valid for all u)

3. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$ (Valid for $|u| > a > 0$)

The derivative formulas in Table 7.3 have $a = 1$, but in most integrations $a \neq 1$, and the formulas in Table 7.4 are more useful.

EXAMPLE 11 Using the Integral Formulas

(a) $$\int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \Big]_{\sqrt{2}/2}^{\sqrt{3}/2}$$

$$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

(b) $$\int_0^1 \frac{dx}{1+x^2} = \tan^{-1} x \Big]_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

(c) $$\int_{2/\sqrt{3}}^{\sqrt{2}} \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x \Big]_{2/\sqrt{3}}^{\sqrt{2}} = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$$

EXAMPLE 12 Using Substitution and Table 7.4

(a) $$\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{dx}{\sqrt{(3)^2-x^2}} = \sin^{-1}\left(\frac{x}{3}\right) + C$$ Table 7.4 Formula 1, with $a = 3, u = x$

(b) $$\int \frac{dx}{\sqrt{3-4x^2}} = \frac{1}{2}\int \frac{du}{\sqrt{a^2-u^2}}$$ $a = \sqrt{3}, u = 2x$, and $du/2 = dx$

$$= \frac{1}{2}\sin^{-1}\left(\frac{u}{a}\right) + C$$ Formula 1

$$= \frac{1}{2}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C$$

EXAMPLE 13 Evaluate $\displaystyle\int \frac{dx}{\sqrt{4x - x^2}}$.

Solution The expression $\sqrt{4x - x^2}$ does not match any of the formulas in Table 7.4, so we first rewrite $4x - x^2$ by completing the square:

$$4x - x^2 = -(x^2 - 4x) = -(x^2 - 4x + 4) + 4 = 4 - (x - 2)^2.$$

Then we substitute $a = 2$, $u = x - 2$, and $du = dx$ to get

$$\int \frac{dx}{\sqrt{4x - x^2}} = \int \frac{dx}{\sqrt{4 - (x - 2)^2}}$$

$$= \int \frac{du}{\sqrt{a^2 - u^2}} \qquad a = 2,\ u = x - 2, \text{ and } du = dx$$

$$= \sin^{-1}\left(\frac{u}{a}\right) + C \qquad \text{Table 7.4, Formula 1}$$

$$= \sin^{-1}\left(\frac{x - 2}{2}\right) + C$$

■

EXAMPLE 14

Evaluate $\displaystyle\int \frac{dx}{4x^2 + 4x + 2}$.

Solution We complete the square on the binomial $4x^2 + 4x$:

$$4x^2 + 4x + 2 = 4(x^2 + x) + 2 = 4\left(x^2 + x + \frac{1}{4}\right) + 2 - \frac{4}{4}$$

$$= 4\left(x + \frac{1}{2}\right)^2 + 1 = (2x + 1)^2 + 1.$$

Then,

$$\int \frac{dx}{4x^2 + 4x + 2} = \int \frac{dx}{(2x + 1)^2 + 1} = \frac{1}{2}\int \frac{du}{u^2 + a^2} \qquad a = 1, u = 2x + 1, \text{ and } du/2 = dx$$

$$= \frac{1}{2} \cdot \frac{1}{a} \tan^{-1}\left(\frac{u}{a}\right) + C \qquad \text{Table 7.4, Formula 2}$$

$$= \frac{1}{2} \tan^{-1}(2x + 1) + C \qquad a = 1, u = 2x + 1$$

EXAMPLE 15 Using Substitution

Evaluate $$\int \frac{dx}{\sqrt{e^{2x} - 6}}.$$

Solution

$$\int \frac{dx}{\sqrt{e^{2x} - 6}} = \int \frac{du/u}{\sqrt{u^2 - a^2}}$$

$u = e^x$, $du = e^x\,dx$, $dx = du/e^x = du/u$, $a = \sqrt{6}$

$$= \int \frac{du}{u\sqrt{u^2 - a^2}}$$

$$= \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$$

Table 7.4, Formula 3

$$= \frac{1}{\sqrt{6}}\sec^{-1}\left(\frac{e^x}{\sqrt{6}}\right) + C$$

EXERCISES 9.5

1. Use reference triangles like those in Examples 1–3 to find the angles in Exercises 1–12.

1. a. $\tan^{-1} 1$ b. $\tan^{-1}(-\sqrt{3})$ c. $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

5. a. $\cos^{-1}\left(\frac{1}{2}\right)$ b. $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ c. $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

8. a. $\sec^{-1}\sqrt{2}$ b. $\sec^{-1}\left(\frac{-2}{\sqrt{3}}\right)$ c. $\sec^{-1} 2$

12. a. $\cot^{-1}(1)$ b. $\cot^{-1}(-\sqrt{3})$ c. $\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)$

2. Trigonometric Function Values

13. Given that $\alpha = \sin^{-1}(5/13)$, find $\cos\alpha$, $\tan\alpha$, $\sec\alpha$, $\csc\alpha$, and $\cot\alpha$.

16. Given that $\alpha = \sec^{-1}(-\sqrt{13}/2)$, find $\sin\alpha$, $\cos\alpha$, $\tan\alpha$, $\csc\alpha$, and $\cot\alpha$.

3. Find the values in Exercises 17–20.

17. $\sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$

18. $\sec\left(\cos^{-1}\frac{1}{2}\right)$

19. $\tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$

20. $\cot\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

4. Finding Trigonometric Expressions, Evaluate the expressions in Exercises 29–37.

29. $\sec\left(\tan^{-1}\frac{x}{2}\right)$

30. $\sec\left(\tan^{-1}2x\right)$

36. $\sin\left(\tan^{-1}\frac{x}{\sqrt{x^2+1}}\right)$

37. $\cos\left(\sin^{-1}\frac{2y}{3}\right)$

5. In Exercises 49–69, find the derivative of *y with respect to the appropriate* variable.

49. $y = \cos^{-1}(x^2)$

50. $y = \cos^{-1}(1/x)$

61. $y = \ln(\tan^{-1}x)$

62. $y = \tan^{-1}(\ln x)$

68. $y = \cot^{-1}\frac{1}{x} - \tan^{-1}x$

69. $y = x\sin^{-1}x + \sqrt{1-x^2}$

6. Evaluate the integrals in Exercises 71–97.

71. $\int \frac{dx}{\sqrt{9-x^2}}$

72. $\int \frac{dx}{\sqrt{1-4x^2}}$

79. $\int_0^2 \frac{dt}{8+2t^2}$

89. $\int_{-\pi/2}^{\pi/2} \frac{2\cos\theta\, d\theta}{1+(\sin\theta)^2}$

90. $\int_{\pi/6}^{\pi/4} \frac{\csc^2 x\, dx}{1+(\cot x)^2}$

97. $\int_{-1}^{0} \frac{6\, dt}{\sqrt{3-2t-t^2}}$

Hyperbolic Functions

The hyperbolic functions are formed by taking combinations of the two exponential functions e^x and e^{-x}. The hyperbolic functions simplify many mathematical expressions and they are important in applications.

In this section,we give a brief introduction to hyperbolic functions, their graphs, how their derivatives are calculated.

˝Even and Odd Parts of the Exponential Function

An even function *f satisfies* f(*-x) = f(x), while an odd function satisfies* f(*-x) = -f(x).*

$$f(x) = \underbrace{\frac{f(x) + f(-x)}{2}}_{\text{even part}} + \underbrace{\frac{f(x) - f(-x)}{2}}_{\text{odd part}}.$$

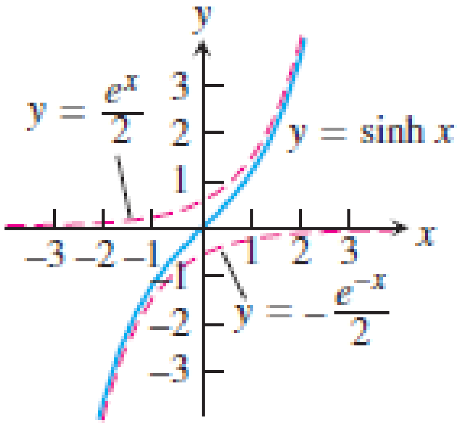
If we write e^x this way, we get

$$e^x = \underbrace{\frac{e^x + e^{-x}}{2}}_{\text{even part}} + \underbrace{\frac{e^x - e^{-x}}{2}}_{\text{odd part}}.$$

Definitions and Identities

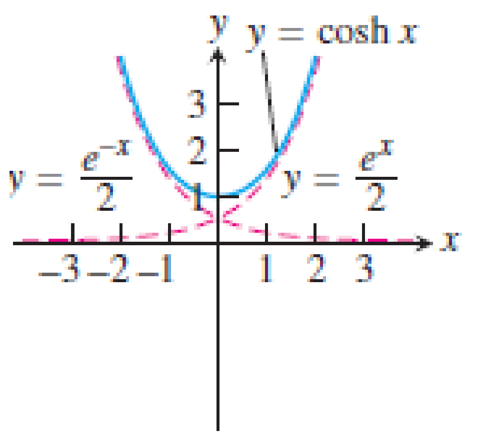
The six basic hyperbolic functions.

Hyperbolic sine of x: $$\sinh x = \frac{e^x - e^{-x}}{2}$$



(a)

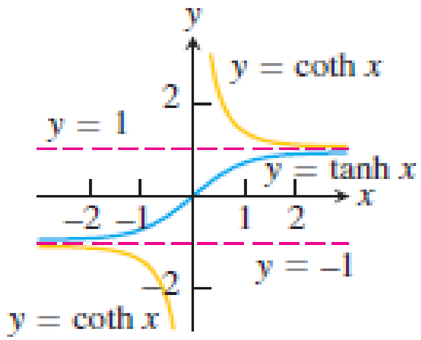
Hyperbolic cosine of x: $$\cosh x = \frac{e^x + e^{-x}}{2}$$



(b)

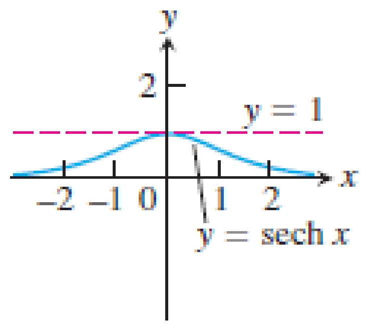
Hyperbolic tangent: $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Hyperbolic cotangent: $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$



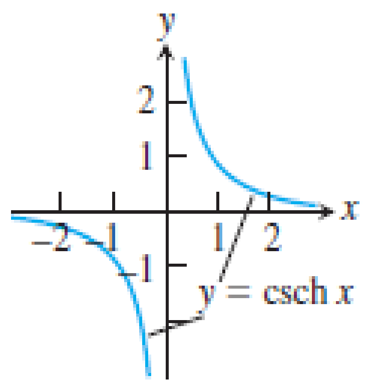
(c)

Hyperbolic secant: $\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$



(d)

Hyperbolic cosecant: $\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$



(e)

Identities for hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$

$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$

$$\tanh^2 x = 1 - \operatorname{sech}^2 x$$

$$\coth^2 x = 1 + \operatorname{csch}^2 x$$

Derivatives and Integrals

The six hyperbolic functions, being rational combinations of the differentiable functions e^x and e^{-x}, have derivatives at every point at which they are defined (Table 7.7). Again, there are similarities with trigonometric functions. The derivative formulas in Table 7.7 lead to the integral formulas in Table 7.8.

TABLE 7.7 Derivatives of hyperbolic functions

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$$

$$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$$

$$\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{du}{dx}$$

TABLE 7.8 Integral formulas for hyperbolic functions

$$\int \sinh u\, du = \cosh u + C$$

$$\int \cosh u\, du = \sinh u + C$$

$$\int \operatorname{sech}^2 u\, du = \tanh u + C$$

$$\int \operatorname{csch}^2 u\, du = -\coth u + C$$

$$\int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C$$

$$\int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C$$

EXAMPLE 1 Finding Derivatives and Integrals

(a) $$\frac{d}{dt}\left(\tanh \sqrt{1+t^2}\right) = \operatorname{sech}^2 \sqrt{1 = t^2} \cdot \frac{d}{dt}\left(\sqrt{1+t^2}\right)$$

$$= \frac{t}{\sqrt{1+t^2}} \operatorname{sech}^2 \sqrt{1+t^2}$$

(b) $$\int \coth 5x\,dx = \int \frac{\cosh 5x}{\sinh 5x}\,dx = \frac{1}{5}\int \frac{du}{u}$$

$u = \sinh 5x,$
$du = 5 \cosh 5x\,dx$

$$= \frac{1}{5}\ln|u| + C = \frac{1}{5}\ln|\sinh 5x| + C$$

(c) $$\int_0^1 \sinh^2 x\,dx = \int_0^1 \frac{\cosh 2x - 1}{2}\,dx$$

Table 7.6

$$= \frac{1}{2}\int_0^1 (\cosh 2x - 1)\,dx = \frac{1}{2}\left[\frac{\sinh 2x}{2} - x\right]_0^1$$

$$= \frac{\sinh 2}{4} - \frac{1}{2} \approx 0.40672$$

Evaluate with a calculator

(d) $$\int_0^{\ln 2} 4e^x \sinh x\,dx = \int_0^{\ln 2} 4e^x \frac{e^x - e^{-x}}{2}\,dx = \int_0^{\ln 2} (2e^{2x} - 2)\,dx$$

$$= \left[e^{2x} - 2x\right]_0^{\ln 2} = (e^{2\ln 2} - 2\ln 2) - (1 - 0)$$

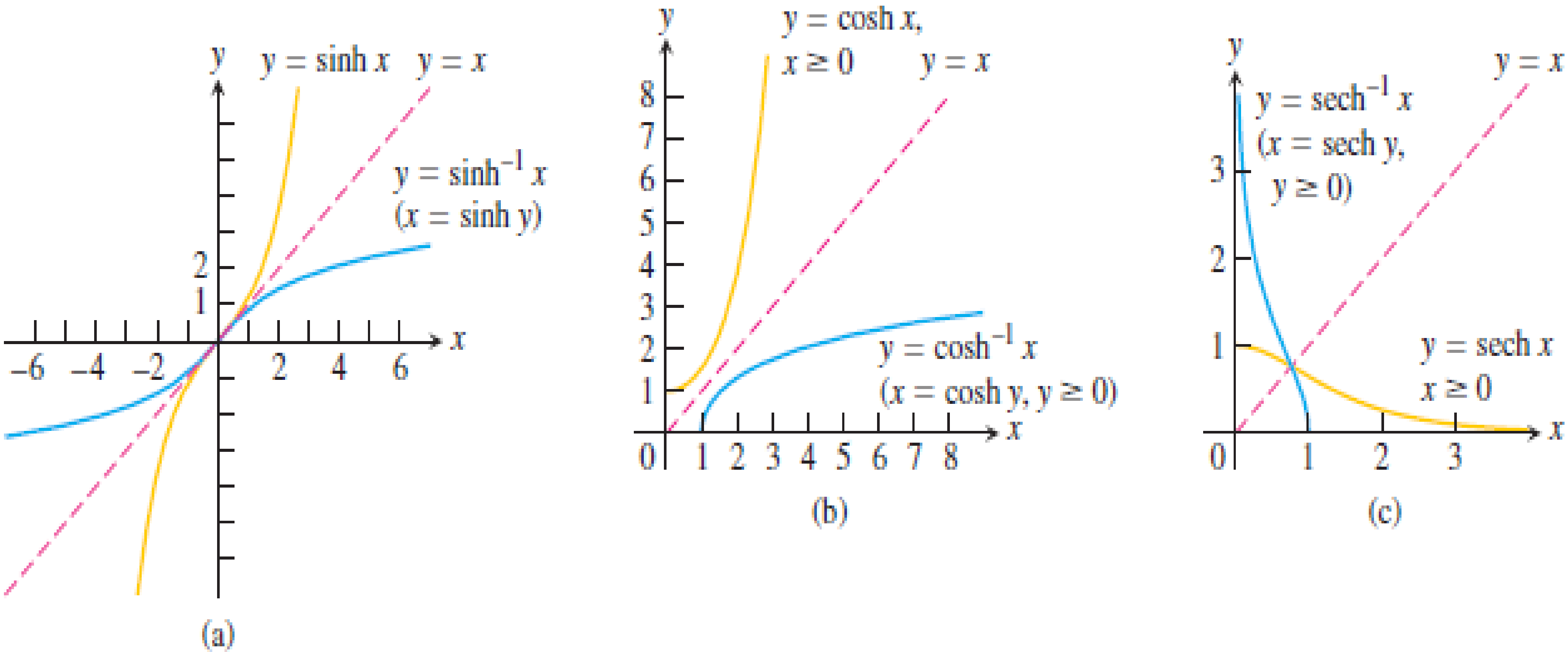
$$= 4 - 2\ln 2 - 1$$

$$\approx 1.6137$$

Inverse Hyperbolic Functions

The inverses of the six basic hyperbolic functions are very useful in integration, We denote its inverse by

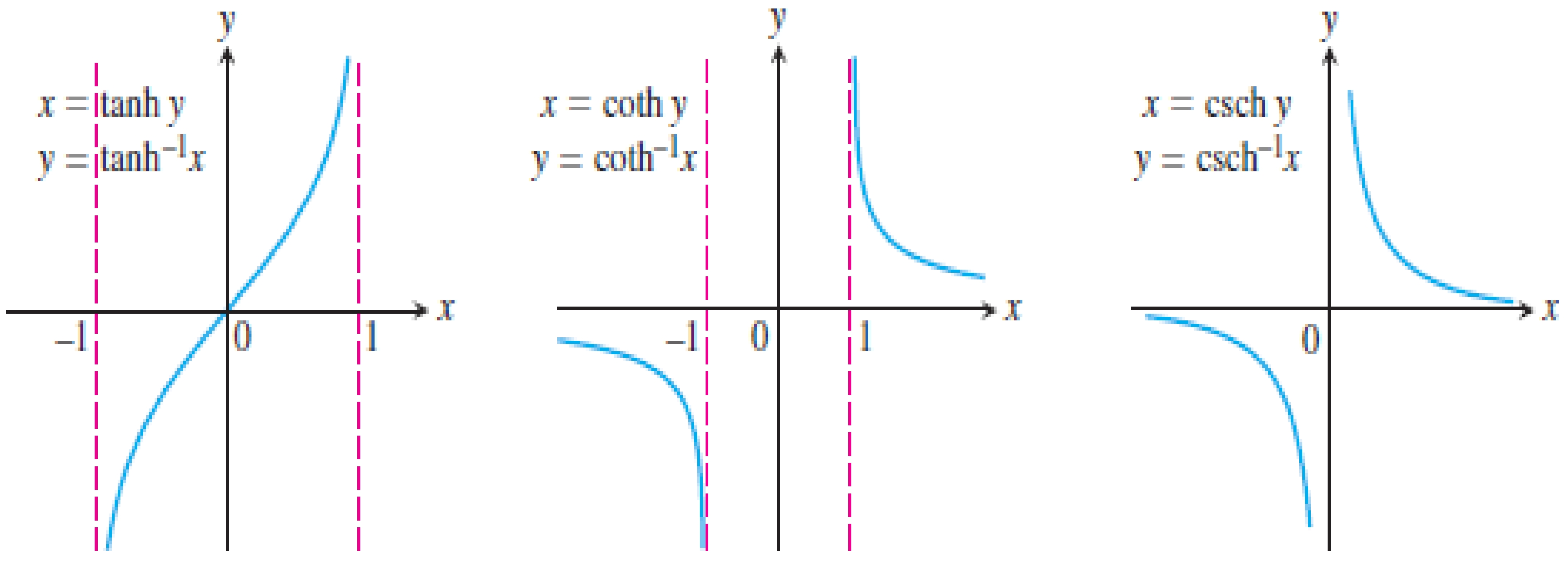
$y = \sinh^{-1} x.$ $y = \cosh^{-1} x.$ $y = \operatorname{sech}^{-1} x.$



The hyperbolic tangent, cotangent, and cosecant are one-to-one on their domains and therefore have inverses, denoted by

$$y = \tanh^{-1} x, \qquad y = \coth^{-1} x, \qquad y = \operatorname{csch}^{-1} x.$$

These functions are graphed in Figure



(a) (b) (c)

Useful Identities for inverse hyperbolic functions

$$\operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x}$$

$$\operatorname{csch}^{-1} x = \sinh^{-1} \frac{1}{x}$$

$$\coth^{-1} x = \tanh^{-1} \frac{1}{x}$$

Derivatives of inverse hyperbolic functions

$$\frac{d(\sinh^{-1} u)}{dx} = \frac{1}{\sqrt{1 + u^2}} \frac{du}{dx}$$

$$\frac{d(\cosh^{-1} u)}{dx} = \frac{1}{\sqrt{u^2 - 1}} \frac{du}{dx}, \qquad u > 1$$

$$\frac{d(\tanh^{-1} u)}{dx} = \frac{1}{1 - u^2} \frac{du}{dx}, \qquad |u| < 1$$

$$\frac{d(\coth^{-1} u)}{dx} = \frac{1}{1 - u^2} \frac{du}{dx}, \qquad |u| > 1$$

$$\frac{d(\operatorname{sech}^{-1} u)}{dx} = \frac{-du/dx}{u\sqrt{1 - u^2}}, \qquad 0 < u < 1$$

$$\frac{d(\operatorname{csch}^{-1} u)}{dx} = \frac{-du/dx}{|u|\sqrt{1 + u^2}}, \qquad u \neq 0$$

EXAMPLE 2: Derivative of the Inverse Hyperbolic Cosine

Show that if *u is a differentiable function of x whose values are greater than 1,then*

$$\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2 - 1}} \frac{du}{dx}.$$

Solution First we find the derivative of $y = \cosh^{-1} x$ for $x > 1$ by applying Theorem 1 with $f(x) = \cosh x$ and $f^{-1}(x) = \cosh^{-1} x$. Theorem 1 can be applied because the derivative of $\cosh x$ is positive for $0 < x$.

$$\begin{aligned}
(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} && \text{Theorem 1} \\
&= \frac{1}{\sinh(\cosh^{-1} x)} && f'(u) = \sinh u \\
&= \frac{1}{\sqrt{\cosh^2(\cosh^{-1} x) - 1}} && \cosh^2 u - \sinh^2 u = 1, \\
& && \sinh u = \sqrt{\cosh^2 u - 1} \\
&= \frac{1}{\sqrt{x^2 - 1}} && \cosh(\cosh^{-1} x) = x
\end{aligned}$$

In short,

$$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2 - 1}}.$$

Integrals leading to inverse hyperbolic functions

1. $\displaystyle\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C, \qquad a > 0$

2. $\displaystyle\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C, \qquad u > a > 0$

3. $\displaystyle\int \frac{du}{a^2-u^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right) + C & \text{if } u^2 < a^2 \\ \frac{1}{a}\coth^{-1}\left(\frac{u}{a}\right) + C, & \text{if } u^2 > a^2 \end{cases}$

4. $\displaystyle\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right) + C, \qquad 0 < u < a$

5. $\displaystyle\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right| + C, \qquad u \neq 0 \text{ and } a > 0$

EXAMPLE 3: Evaluate $\displaystyle\int_0^1 \frac{2\,dx}{\sqrt{3+4x^2}}.$

Solution The indefinite integral is

$$\int \frac{2\,dx}{\sqrt{3+4x^2}} = \int \frac{du}{\sqrt{a^2+u^2}} \qquad u = 2x,\quad du = 2\,dx,\quad a = \sqrt{3}$$

$$= \sinh^{-1}\left(\frac{u}{a}\right) + C \qquad \text{Formula from Table 7.11}$$

$$= \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C.$$

Therefore,

$$\int_0^1 \frac{2\,dx}{\sqrt{3+4x^2}} = \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right)\Bigg]_0^1 = \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - \sinh^{-1}(0)$$

$$= \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - 0 \approx 0.98665.$$

EXERCISES 9.6

1. Each of Exercises 1–4 gives a value of $\sinh x$ or $\cosh x$. Use the definitions and the identity $\cosh^2 x - \sinh^2 x = 1$ to find the values of the remaining five hyperbolic functions.

1. $\sinh x = -\frac{3}{4}$
2. $\sinh x = \frac{4}{3}$
3. $\cosh x = \frac{17}{15}, \quad x > 0$
4. $\cosh x = \frac{13}{5}, \quad x > 0$

2. In Exercises 13–26, find the derivative of *y with respect to the appropriate* variable .

13. $y = 6 \sinh \frac{x}{3}$
14. $y = \frac{1}{2} \sinh (2x + 1)$
25. $y = \sinh^{-1} \sqrt{x}$
26. $y = \cosh^{-1} 2\sqrt{x + 1}$

4. Evaluate the integrals in Exercises 41–53.

41. $\int \sinh 2x \, dx$

53. $\int_{-\ln 4}^{-\ln 2} 2e^{\theta} \cosh \theta \, d\theta$

47. $\int \operatorname{sech}^2 \left(x - \frac{1}{2}\right) dx$

48. $\int \operatorname{csch}^2 (5 - x) \, dx$

5. Evaluate the integrals in Exercises 67–71 in terms of

a. inverse hyperbolic functions. **b. natural logarithms.**

67. $\int_0^{2\sqrt{3}} \frac{dx}{\sqrt{4 + x^2}}$

69. $\int_{5/4}^{2} \frac{dx}{1 - x^2}$

71. $\int_{1/5}^{3/13} \frac{dx}{x\sqrt{1 - 16x^2}}$