MATHEMATICS II

SECOND SEMESTER

Lec. 08

INTEGRATION

Outlines

- Indefinite Integral
- Basic Integration Formulas
- The Substitution Rule
- The Integrals of Sin^2x and Cos^2x
- Definite Integral
- Area Under a Curve as Definite Integral

Indefinite Integrals and the Substitution Rule

DEFINITION Indefinite Integral, Integrand

The set of all antiderivatives of f is the **indefinite integral** of f with respect to x, denoted by

$$\int f(x)\,dx.$$

The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration.**

EXAMPLE 1: Evaluate $\int (x^2 - 2x + 5)\,dx.$

Solution:

$$\int (x^2 - 2x + 5)\,dx = \int x^2\,dx - \int 2x\,dx + \int 5\,dx$$

$$= \int x^2\,dx - 2\int x\,dx + 5\int 1\,dx$$

$$= \left(\frac{x^3}{3} + C_1\right) - 2\left(\frac{x^2}{2} + C_2\right) + 5(x + C_3)$$

$$= \frac{x^3}{3} + C_1 - x^2 - 2C_2 + 5x + 5C_3.$$

This formula is more complicated than it needs to be. If we combine C_1, $-2C_2$, and $5C_3$ into a single arbitrary constant $C = C_1 - 2C_2 + 5C_3$, the formula simplifies to

$$\frac{x^3}{3} - x^2 + 5x + C$$

Basic Integration Formulas

1. $\int du = u + c$, $\int (du + dv) = \int du + \int dv$, and $\int k du = ku + c$

2. $\int u^n \, du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$

3. $\int \frac{du}{u} = \ln |u| + C$

4. $\int \sin u \, du = -\cos u + C$

5. $\int \cos u \, du = \sin u + C$

6. $\int \tan u \, du = \ln |\sec u| + C$

7. $\int \cot u \, du = \ln|\sin u| + C$

8. $\int \sec u \, du = \ln|\sec u + \tan u| + C$

9. $\int \csc u \, du = \ln|\csc u - \cot u| + C$

10. $\int \sec^2 u \, du = \tan u + C$

11. $\int \csc^2 u \, du = -\cot u + C$

12. $\int \sec u \tan u \, du = \sec u + C$

13. $\int \csc u \cot u \, du = -\csc u + C$

Solved Problems

In Problems 1 to 8, evaluate the indefinite integral at the left.

1. $$\int x^5\,dx = \frac{x^6}{6} + C$$

2. $$\int \frac{dx}{x^2} = \int x^{-2}\,dx = \frac{x^{-1}}{-1} + C = -\frac{1}{x} + C$$

3. $$\int \sqrt[3]{z}\,dz = \int z^{1/3}\,dz = \frac{z^{4/3}}{4/3} + C = \frac{3}{4}z^{4/3} + C$$

4. $$\int \frac{dx}{\sqrt[3]{x^2}} = \int x^{-2/3}\,dx = \frac{x^{1/3}}{1/3} + C = 3x^{1/3} + C$$

5. $$\int (2x^2 - 5x + 3)\,dx = 2\int x^2\,dx - 5\int x\,dx + 3\int dx = \frac{2x^3}{3} - \frac{5x^2}{2} + 3x + C$$

6. $$\int (1-x)\sqrt{x}\,dx = \int (x^{1/2} - x^{3/2})\,dx = \int x^{1/2}\,dx - \int x^{3/2}\,dx = \tfrac{2}{3}x^{3/2} - \tfrac{2}{5}x^{5/2} + C$$

7. $$\int (3s+4)^2\,ds = \int (9s^2 + 24s + 16)\,ds = 9(\tfrac{1}{3}s^3) + 24(\tfrac{1}{2}s^2) + 16s + C = 3s^3 + 12s^2 + 16s + C$$

8. $$\int \frac{x^3 + 5x^2 - 4}{x^2}\,dx = \int (x + 5 - 4x^{-2})\,dx = \frac{1}{2}x^2 + 5x - \frac{4x^{-1}}{-1} + C = \frac{1}{2}x^2 + 5x + \frac{4}{x} + C$$

”The Substitution Rule

Substitution: Running the Chain Rule Backwards

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

The Substitution Rule provides the following method to evaluate the integral

$$\int f(g(x))g'(x)\,dx,$$

when f and g' are continuous functions:

1. Substitute $u = g(x)$ and $du = g'(x)\,dx$ to obtain the integral

$$\int f(u)\,du.$$

2. Integrate with respect to u.
3. Replace u by $g(x)$ in the result.

To evaluate an antiderivative $\int f(x)\,dx$, it is often useful to

replace x with a new variable u by means of a *substitution* $x = g(u)$, $dx = g'(u)\,du$. The equation

$$\int f(x)\,dx = \int f(g(u))g'(u)\,du$$

EXAMPLE 1:

To evaluate $\int (x+3)^{11}\,dx$, replace $x+3$ with u; that is, let $x = u - 3$. Then $dx = du$, and we obtain

$$\int (x+3)^{11}\,dx = \int u^{11}\,du = \tfrac{1}{12}u^{12} + C = \tfrac{1}{12}(x+3)^{12} + C$$

EXAMPLE 1: Using the Power Rule

$$\int \sqrt{1 + y^2} \cdot 2y \, dy = \int \sqrt{u} \cdot \left(\frac{du}{dy}\right) dy$$

Let $u = 1 + y^2$,
$du/dy = 2y$

$$= \int u^{1/2} \, du$$

$$= \frac{u^{(1/2)+1}}{(1/2) + 1} + C$$

Integrate, using Eq. (1) with $n = 1/2$.

$$= \frac{2}{3} u^{3/2} + C$$

Simpler form

$$= \frac{2}{3}(1 + y^2)^{3/2} + C$$

Replace u by $1 + y^2$.

EXAMPLE 2 : Adjusting the Integrand by a Constant

$$\int \sqrt{4t-1}\,dt = \int \frac{1}{4}\cdot\sqrt{4t-1}\cdot 4\,dt$$

$$= \frac{1}{4}\int \sqrt{u}\cdot\left(\frac{du}{dt}\right)dt$$ Let $u = 4t - 1$, $du/dt = 4$.

$$= \frac{1}{4}\int u^{1/2}\,du$$ With the 1/4 out front, the integral is now in standard form.

$$= \frac{1}{4}\cdot\frac{u^{3/2}}{3/2} + C$$ Integrate, using Eq. (1) with $n = 1/2$.

$$= \frac{1}{6}u^{3/2} + C$$ Simpler form

$$= \frac{1}{6}(4t-1)^{3/2} + C$$ Replace u by $4t - 1$.

EXAMPLE 3: Using Substitution

$$\int \cos(7\theta + 5)\, d\theta = \int \cos u \cdot \frac{1}{7}\, du$$

Let $u = 7\theta + 5$, $du = 7\, d\theta$, $(1/7)\, du = d\theta$.

$$= \frac{1}{7} \int \cos u\, du$$

With the $(1/7)$ out front, the integral is now in standard form.

$$= \frac{1}{7} \sin u + C$$

Integrate with respect to u, Table 4.2.

$$= \frac{1}{7} \sin(7\theta + 5) + C$$

Replace u by $7\theta + 5$.

EXAMPLE 4: Using Substitution

$$\int x^2 \sin(x^3)\, dx = \int \sin(x^3) \cdot x^2\, dx$$

$$= \int \sin u \cdot \frac{1}{3}\, du$$

Let $u = x^3$,
$du = 3x^2\, dx$,
$(1/3)\, du = x^2\, dx$.

$$= \frac{1}{3} \int \sin u\, du$$

$$= \frac{1}{3}(-\cos u) + C$$

Integrate with respect to u.

$$= -\frac{1}{3} \cos(x^3) + C$$

Replace u by x^3.

EXAMPLE 6: Using Different Substitutions, Evaluate $\int \frac{2z\,dz}{\sqrt[3]{z^2+1}}$.

Solution 1: Substitute $u = z^2 + 1$.

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{du}{u^{1/3}}$$ Let $u = z^2 + 1$, $du = 2z\,dz$.

$$= \int u^{-1/3}\,du$$ In the form $\int u^n\,du$

$$= \frac{u^{2/3}}{2/3} + C$$ Integrate with respect to u.

$$= \frac{3}{2}u^{2/3} + C$$

$$= \frac{3}{2}(z^2+1)^{2/3} + C$$ Replace u by $z^2 + 1$.

Solution 2: Substitute $u = \sqrt[3]{z^2+1}$ instead.

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{3u^2\,du}{u}$$ Let $u = \sqrt[3]{z^2+1}$, $u^3 = z^2 + 1$, $3u^2\,du = 2z\,dz$.

$$= 3\int u\,du$$

$$= 3 \cdot \frac{u^2}{2} + C$$ Integrate with respect to u.

$$= \frac{3}{2}(z^2+1)^{2/3} + C$$ Replace u by $(z^2+1)^{1/3}$.

The integrals of sin^2x and cos^2x

EXAMPLE 7

(a) $$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx \qquad \sin^2 x = \frac{1-\cos 2x}{2}$$

$$= \frac{1}{2}\int (1-\cos 2x)\,dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\,dx$$

$$= \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + C = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

(b) $$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

$$= \frac{x}{2} + \frac{\sin 2x}{4} + C$$

As in part (a), but with a sign change

Note: integration, **cos kx =** $\frac{sin\ kx}{k}$ **and sin kx = -** $\frac{cos\ kx}{k}$

EXERCISES 8.1

1. Evaluate the indefinite integrals in Exercises 1–4 by using the given substitutions to reduce the integrals to standard form.

1. $\int \sin 3x \, dx, \quad u = 3x$

2. $\int x^3(x^4 - 1)^2 \, dx, \quad u = x^4 - 1$

3. $\int \sec 2t \tan 2t \, dt, \quad u = 2t$

4. $\int \frac{dx}{\sqrt{5x + 8}}$

a. Using $u = 5x + 8$ b. Using $u = \sqrt{5x + 8}$

2. Evaluate the integrals :

1. $\int (2x + 1)^3 \, dx$

2. $\int \frac{3 \, dx}{(2 - x)^2}$

3. $\int \theta \sqrt[4]{1 - \theta^2} \, d\theta$

4. $\int \cos(3z + 4) \, dz$

5. $\int \frac{4y \, dy}{\sqrt{2y^2 + 1}}$

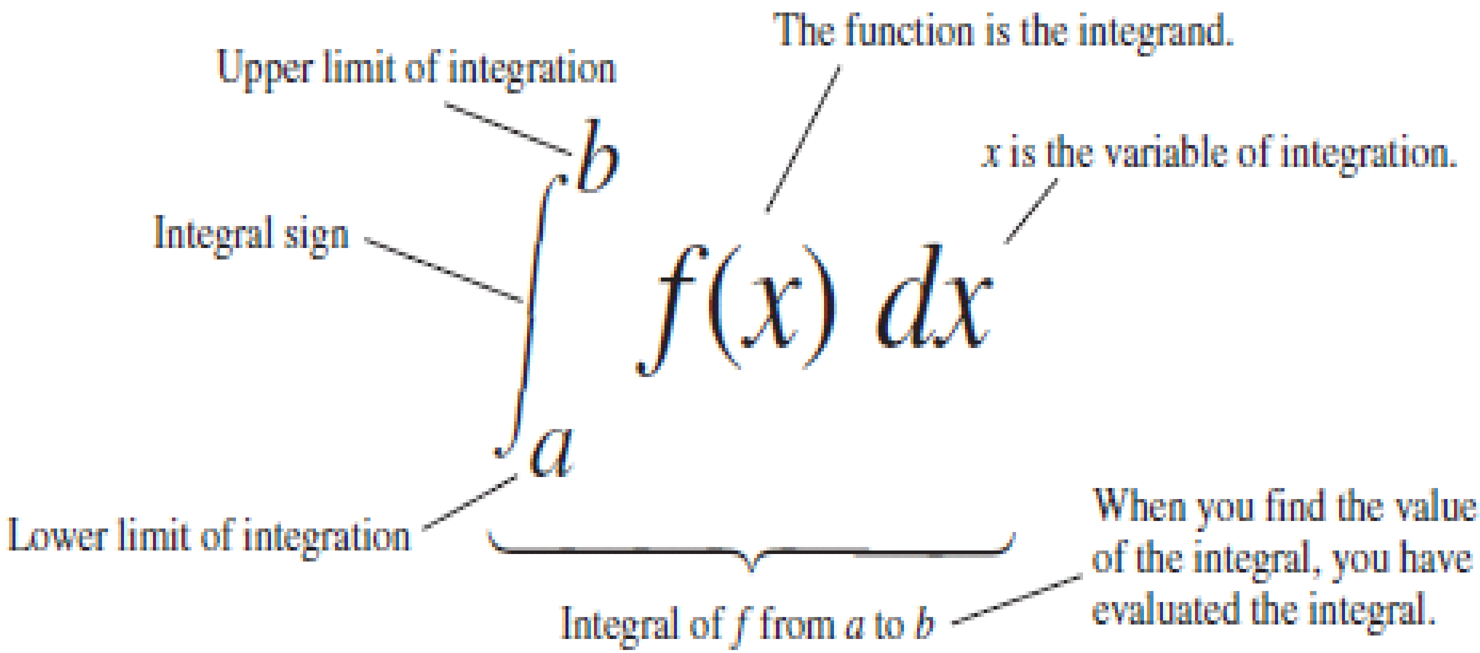
6. $\int \sec^2(3x + 2) \, dx$

”The Definite Integral

The symbol for the number I in the definition of the definite integral is

$$\int_a^b f(x)\,dx$$

which is read as “the integral from a to b of f of x dee x” or sometimes as “the integral from a to b of f of x with respect to x.” The component parts in the integral symbol also have names:



Properties of Definite Integrals

When f and g are integrable on the interval [a , b], the definite integral satisfies Rules 1 to 7 in Table 5.1.

Table 5.1 Rules satisfied by definite integrals

1.	*Order of Integration:*	$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$	A Definition
2.	*Zero Width Interval:*	$\int_a^a f(x)\,dx = 0$	Also a Definition
3.	*Constant Multiple:*	$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$	Any Number k
		$\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$	$k = -1$

4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x))\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$

5. *Additivity:* $\int_a^b f(x)\, dx + \int_b^c f(x)\, dx = \int_a^c f(x)\, dx$

6. *Max-Min Inequality:* If f has maximum value $\max f$ and minimum value $\min f$ on $[a, b]$, then

$$\min f \cdot (b - a) \le \int_a^b f(x)\, dx \le \max f \cdot (b - a).$$

7. *Domination:* $f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\, dx \ge \int_a^b g(x)\, dx$

$f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\, dx \ge 0$ (Special Case)

EXAMPLE 1: Using the Rules for Definite Integrals, Suppose that.

$$\int_{-1}^{1} f(x)\,dx = 5, \qquad \int_{1}^{4} f(x)\,dx = -2, \qquad \int_{-1}^{1} h(x)\,dx = 7.$$

Then

1. $\int_{4}^{1} f(x)\,dx = -\int_{1}^{4} f(x)\,dx = -(-2) = 2$ — Rule 1

2. $\int_{-1}^{1} [2f(x) + 3h(x)]\,dx = 2\int_{-1}^{1} f(x)\,dx + 3\int_{-1}^{1} h(x)\,dx$ — Rules 3 and 4

$$= 2(5) + 3(7) = 31$$

3. $\int_{-1}^{4} f(x)\,dx = \int_{-1}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx = 5 + (-2) = 3$ — Rule 5

EXAMPLE 2: Finding Bounds for an Integral

Show that the value of $\int_0^1 \sqrt{1+\cos x}\,dx$ is less than 3\2

Solution The Max-Min Inequality for definite integrals (Rule 6) says that $\min f \cdot (b-a)$ is a *lower bound* for the value of $\int_a^b f(x)\,dx$ and that $\max f \cdot (b-a)$ is an *upper bound*. The maximum value of $\sqrt{1+\cos x}$ on $[0, 1]$ is $\sqrt{1+1} = \sqrt{2}$, so

$$\int_0^1 \sqrt{1+\cos x}\,dx \le \sqrt{2}\cdot(1-0) = \sqrt{2}.$$

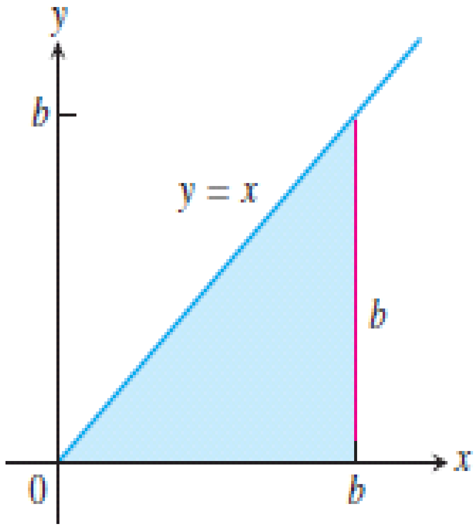
Since $\int_0^1 \sqrt{1+\cos x}\,dx$ is bounded from above by $\sqrt{2}$ (which is 1.414 ...), the integral is less than $3/2$. ■

Area Under a Curve as a Definite Integral

If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the **area under the curve $y = f(x)$ over $[a, b]$** is the integral of f from a to b,

$$A = \int_a^b f(x)\,dx.$$

EXAMPLE 3: Area Under the Line $y = x$, find the area A under $y = x$ over the interval [0,b], b> 0.



Solution: We compute the area in way.

we have that $$\int_0^b x\,dx = \frac{b^2}{2}.$$

Example 4 can be to integrate f(x) = x over any closed interval [a,b], 0< $a < b$.

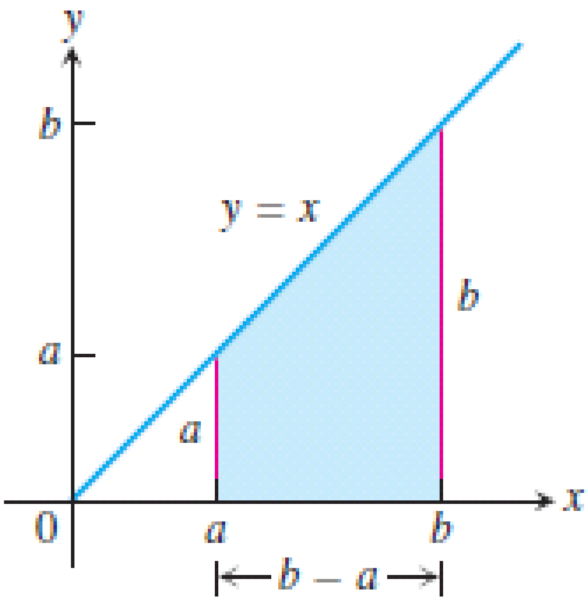
$$\int_a^b x\,dx = \int_a^0 x\,dx + \int_0^b x\,dx \qquad \text{Rule 5}$$

$$= -\int_0^a x\,dx + \int_0^b x\,dx \qquad \text{Rule 1}$$

$$= -\frac{a^2}{2} + \frac{b^2}{2}. \qquad \text{Example 4}$$

In conclusion, we have the following rule for integrating $f(x) = x$.

$$\int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2}, \qquad a < b \qquad (1)$$



This formula gives the area of a trapezoid down to the line $y = x$ (see Figure).

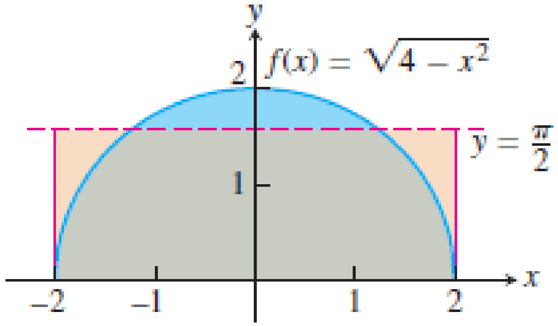
DEFINITION **The Average or Mean Value of a Function**

If f is integrable on $[a, b]$, then its **average value on** $[a, b]$, also called its **mean value**, is

$$\text{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

EXAMPLE 4: Find the average value of $f(x) = \sqrt{4 + x^2}$ on [-2,2].

Solution: $\text{Area} = \frac{1}{2} \cdot \pi r^2 = \frac{1}{2} \cdot \pi(2)^2 = 2\pi.$



Because f is nonnegative, the area is also the value of the integral of f from -2 to 2,

$$\int_{-2}^{2}\sqrt{4-x^2}\,dx = 2\pi.$$

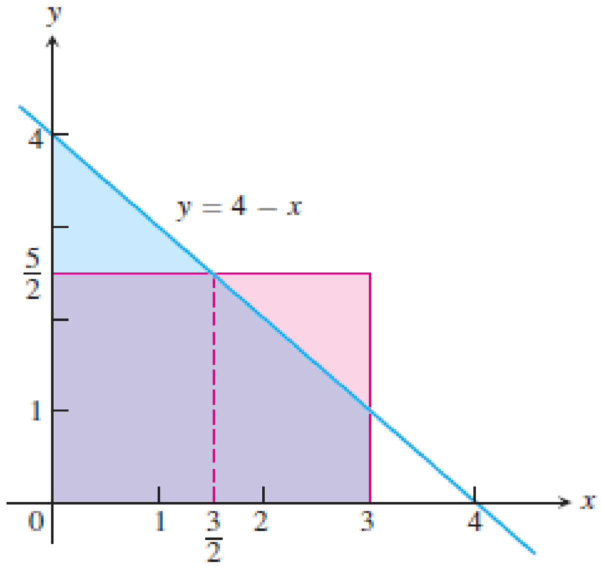
Therefore, the average value of f is

$$\text{av}(f) = \frac{1}{2-(-2)}\int_{-2}^{2}\sqrt{4-x^2}\,dx = \frac{1}{4}(2\pi) = \frac{\pi}{2}.$$

EXAMPEL 5: Find the average value of $f(x) = 4 - x$ on $[0, 3]$

Solution

$$\begin{aligned}
\text{av}(f) &= \frac{1}{b-a}\int_a^b f(x)\,dx \\
&= \frac{1}{3-0}\int_0^3 (4-x)\,dx = \frac{1}{3}\left(\int_0^3 4\,dx - \int_0^3 x\,dx\right) \\
&= \frac{1}{3}\left(4(3-0) - \left(\frac{3^2}{2} - \frac{0^2}{2}\right)\right) \\
&= 4 - \frac{3}{2} = \frac{5}{2}.
\end{aligned}$$



The average value of $f(x) = 4 - x$ over $[0, 3]$ is $5/2$. The function assumes this value when $4 - x = 5/2$ or $x = 3/2$.

EXERCISES 8.2

1. Using Properties and Known Values to Find Other Integrals.

A. Suppose that f and h are integrable and that

$$\int_1^9 f(x)\,dx = -1, \quad \int_7^9 f(x)\,dx = 5, \quad \int_7^9 h(x)\,dx = 4.$$

Use the rules in Table 5.3 to find

a. $\int_1^9 -2f(x)\,dx$

b. $\int_7^9 [f(x) + h(x)]\,dx$

c. $\int_7^9 [2f(x) - 3h(x)]\,dx$

d. $\int_9^1 f(x)\,dx$

e. $\int_1^7 f(x)\,dx$

f. $\int_9^7 [h(x) - f(x)]\,dx$

B. Suppose that $\int_{-3}^0 g(t)\,dt = \sqrt{2}$. Find

a. $\int_0^{-3} g(t)\,dt$

b. $\int_{-3}^0 g(u)\,du$

c. $\int_{-3}^0 [-g(x)]\,dx$

d. $\int_{-3}^0 \frac{g(r)}{\sqrt{2}}\,dr$

2. Use the rules in Table 5.1 and Equations (1) to evaluate the integrals in Exercises 41–50.

1. $\int_0^2 5x\,dx$

2. $\int_3^5 \frac{x}{8}\,dx$

3. $\int_2^1 \left(1 + \frac{z}{2}\right) dz$

4. $\int_3^0 (2z - 3)\,dz$

5. $\int_0^2 (3x^2 + x - 5)\,dx$

6. $\int_1^0 (3x^2 + x - 5)\,dx$

3. In Exercises 1–4 use a definite integral to find the area of the region between the given curve and the *x*-axis on the interval [0, *b*].

1. $y = 3x^2$

2. $y = \pi x^2$

3. $y = 2x$

4. $y = \frac{x}{2} + 1$

4. In Exercises 1– 4, graph the function and find its average value over the given interval.

1. $f(x) = -\frac{x^2}{2}$ on $[0, 3]$ 2. $f(x) = -3x^2 - 1$ on $[0, 1]$

3. $h(x) = -|x|$ on a. $[-1, 0]$, b. $[0, 1]$, and c. $[-1, 1]$

4. $f(t) = (t - 1)^2$ on $[0, 3]$

5. Theory and Examples

1. What values of a and b maximize the value of

$$\int_a^b (x - x^2)\,dx?$$

2. Use the Max-Min Inequality to find upper and lower bounds for the value of

$$\int_0^1 \frac{1}{1 + x^2}\,dx.$$

6. (*Continuation of Exercise 5*) Use the Max-Min Inequality to find upper and lower bounds for

$$\int_0^{0.5} \frac{1}{1+x^2}\,dx \quad \text{and} \quad \int_{0.5}^{1} \frac{1}{1+x^2}\,dx.$$

Add these to arrive at an improved estimate of

$$\int_0^1 \frac{1}{1+x^2}\,dx.$$

7. Show that the value of $\int_0^1 \sin(x^2)\,dx$ cannot possibly be 2.

8. Show that the value of $\int_1^0 \sqrt{x+8}\,dx$ lies between $2\sqrt{2} \approx 2.8$ and 3.

Total Area

When we add up such terms for a negative function we get the negative of the area between the curve and the *x*-axis. If we then take the absolute value, we obtain the correct positive area.

EXAMPLE 5: Calculate the area bounded by the *x*-axis and the parabola $y = 6 - x - x^2$

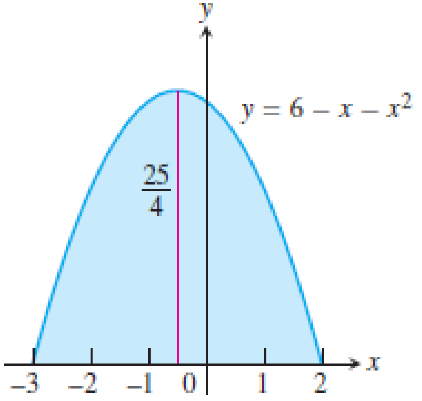
Solution: $y = 0 = 6 - x - x^2 = (3 + x)(2 - x)$, which gives $x = -3$ or $x = 2$.

The curve is sketched in Figure 5.21, and is nonnegative on $[-3, 2]$.

The area is

$$\int_{-3}^{2} (6 - x - x^2)\,dx = \left[6x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-3}^{2}$$

$$= \left(12 - 2 - \frac{8}{3}\right) - \left(-18 - \frac{9}{2} + \frac{27}{3}\right) = 20\tfrac{5}{6}.$$



The curve in Figure 5.21 is an arch of a parabola, and it is interesting to note that the area under such an arch is exactly equal to two-thirds the base times the altitude:

$$\frac{2}{3}(5)\left(\frac{25}{4}\right) = \frac{125}{6} = 20\tfrac{5}{6}.$$

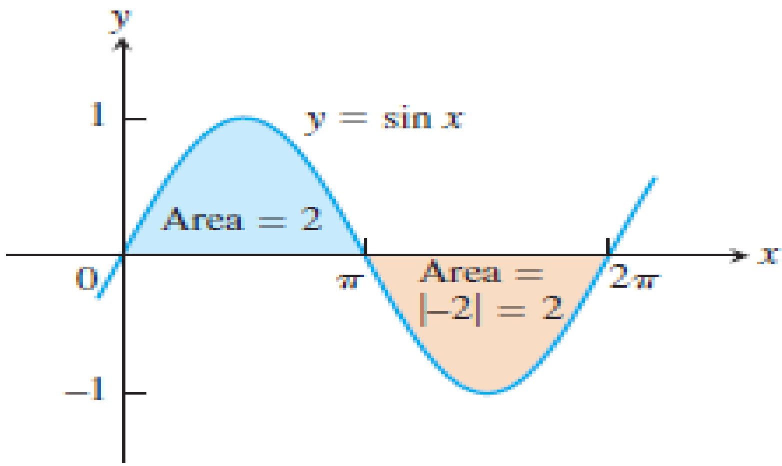
■

EXAMPLE 6:

Figure, shows the graph of the function f(*x*) = sin *x* between *x* = 0 and *x* = 2π Compute

(a) the definite integral of f(*x*) over [0, 2π]

(b) the area between the graph of f(*x*) and the *x*-axis over [0, 2π]



Solution The definite integral for $f(x) = \sin x$ is given by

$$\int_0^{2\pi} \sin x\, dx = -\cos x \Big]_0^{2\pi} = -[\cos 2\pi - \cos 0] = -[1 - 1] = 0.$$

The area between the graph of $f(x)$ and the x-axis over $[0, 2\pi]$ is calculated by breaking up the domain of $\sin x$ into two pieces: the interval $[0, \pi]$ over which it is nonnegative and the interval $[\pi, 2\pi]$ over which it is nonpositive.

$$\int_0^{\pi} \sin x \, dx = -\cos x \Big]_0^{\pi} = -[\cos \pi - \cos 0] = -[-1 - 1] = 2.$$

$$\int_{\pi}^{2\pi} \sin x \, dx = -\cos x \Big]_{\pi}^{2\pi} = -[\cos 2\pi - \cos \pi] = -[1 - (-1)] = -2.$$

The second integral gives a negative value. The area between the graph and the axis is obtained by adding the absolute values

$$\text{Area} = |2| + |-2| = 4.$$

■

Summary:

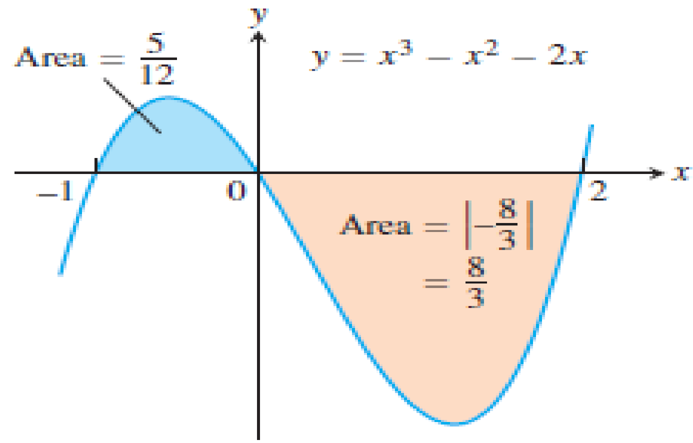
To find the area between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$, do the following:

1. Subdivide $[a, b]$ at the zeros of f.
2. Integrate f over each subinterval.
3. Add the absolute values of the integrals.

EXAMPLE 7:

Find the area of the region between the *x*-axis and the graph of

$f(x) = x3 - x2 - 2x, \qquad -1 \leq x \leq 2.$



Solution:

We integrate f over each subinterval and add the absolute values of the calculated integrals.

$$\int_{-1}^{0}(x^3 - x^2 - 2x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} = 0 - \left[\frac{1}{4} + \frac{1}{3} - 1\right] = \frac{5}{12}$$

$$\int_{0}^{2}(x^3 - x^2 - 2x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{0}^{2} = \left[4 - \frac{8}{3} - 4\right] - 0 = -\frac{8}{3}$$

The total enclosed area is obtained by adding the absolute values of the calculated integrals,

$$\text{Total enclosed area} = \frac{5}{12} + \left|-\frac{8}{3}\right| = \frac{37}{12}.$$

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EXERCISES 8.3:

1. Evaluate the integrals in Exercises.

1. $\int_{-2}^{0} (2x + 5)\, dx$

2. $\int_{-3}^{4} \left(5 - \frac{x}{2}\right) dx$

3. $\int_{0}^{4} \left(3x - \frac{x^3}{4}\right) dx$

4. $\int_{-2}^{2} (x^3 - 2x + 3)\, dx$

9. $\int_{0}^{\pi} \sin x\, dx$

10. $\int_{0}^{\pi} (1 + \cos x)\, dx$

11. $\int_{0}^{\pi/3} 2\sec^2 x\, dx$

12. $\int_{\pi/6}^{5\pi/6} \csc^2 x\, dx$

15. $\int_{\pi/2}^{0} \frac{1 + \cos 2t}{2}\, dt$

16. $\int_{-\pi/3}^{\pi/3} \frac{1 - \cos 2t}{2}\, dt$

17. $\int_{-\pi/2}^{\pi/2} (8y^2 + \sin y)\, dy$

18. $\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \frac{\pi}{t^2}\right) dt$

21. $\int_{\sqrt{2}}^{1} \left(\frac{u^7}{2} - \frac{1}{u^5}\right) du$

22. $\int_{1/2}^{1} \left(\frac{1}{v^3} - \frac{1}{v^4}\right) dv$

23. $\int_{1}^{\sqrt{2}} \frac{s^2 + \sqrt{s}}{s^2}\, ds$

24. $\int_{9}^{4} \frac{1 - \sqrt{u}}{\sqrt{u}}\, du$

2. In Exercises, find the total area between the region and the *x-axis.*

37. $y = -x^2 - 2x, \quad -3 \le x \le 2$

39. $y = x^3 - 3x^2 + 2x, \quad 0 \le x \le 2$

3. Find the areas of the shaded regions in Exercises.

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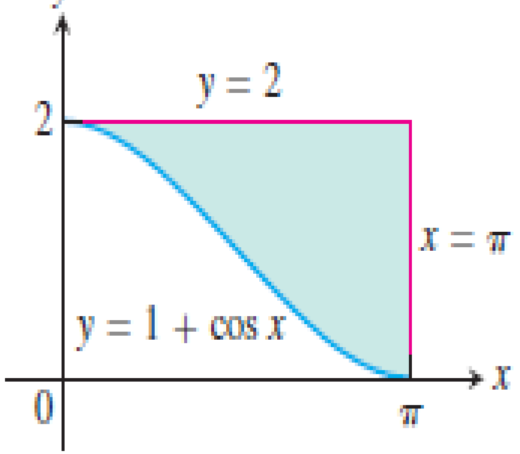

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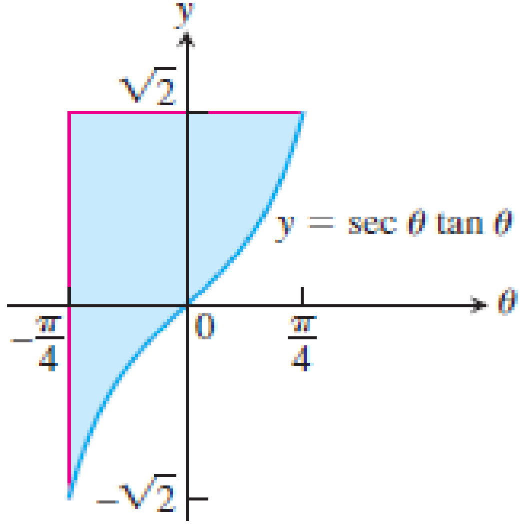

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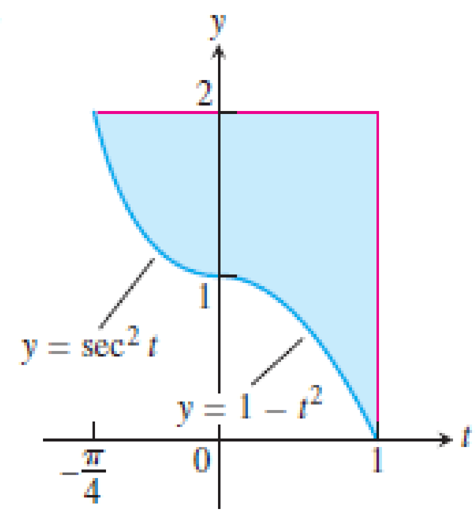


Substitution and Area Between Curves

Substitution in Definite Integrals

If g' is continuous on the interval $[a, b]$ and f is continuous on the range of g, then.

$$\int_a^b f(g(x)) \cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

EXAMPLE 8: Evaluate $\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx.$

Solution We have two choices. $\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$

Method 1:

$$= \int_0^2 \sqrt{u}\,du$$

Let $u = x^3 + 1$, $du = 3x^2\,dx$.
When $x = -1$, $u = (-1)^3 + 1 = 0$.
When $x = 1$, $u = (1)^3 + 1 = 2$.

$$= \frac{2}{3}u^{3/2}\Big]_0^2$$

Evaluate the new definite integral.

$$= \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$

Method 2:

$$\int 3x^2\sqrt{x^3+1}\,dx = \int \sqrt{u}\,du$$ Let $u = x^3 + 1$, $du = 3x^2\,dx$.

$$= \frac{2}{3}u^{3/2} + C$$ Integrate with respect to u.

$$= \frac{2}{3}(x^3+1)^{3/2} + C$$ Replace u by $x^3 + 1$.

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \frac{2}{3}(x^3+1)^{3/2}\Big]_{-1}^{1}$$ Use the integral just found, with limits of integration for x.

$$= \frac{2}{3}\left[((1)^3+1)^{3/2} - ((-1)^3+1)^{3/2}\right]$$

$$= \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$ ■

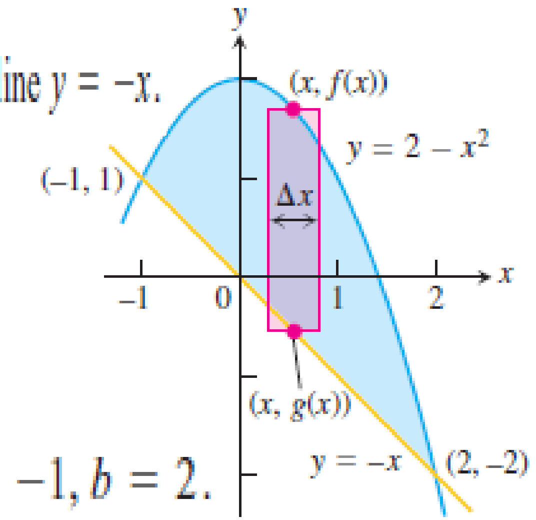
″Area Between Curves

DEFINITION **Area Between Curves**

If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves** $y = f(x)$ **and** $y = g(x)$ **from** a **to** b **is the integral of** $(f - g)$ from a to b:

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

EXAMPLE 9: Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.



Solution

$$2 - x^2 = -x \qquad \text{Equate } f(x) \text{ and } g(x).$$
$$x^2 - x - 2 = 0 \qquad \text{Rewrite.}$$
$$(x + 1)(x - 2) = 0 \qquad \text{Factor.}$$
$$x = -1, \qquad x = 2. \qquad \text{Solve.}$$

The region runs from $x = -1$ to $x = 2$. The limits of integration are $a = -1$, $b = 2$.

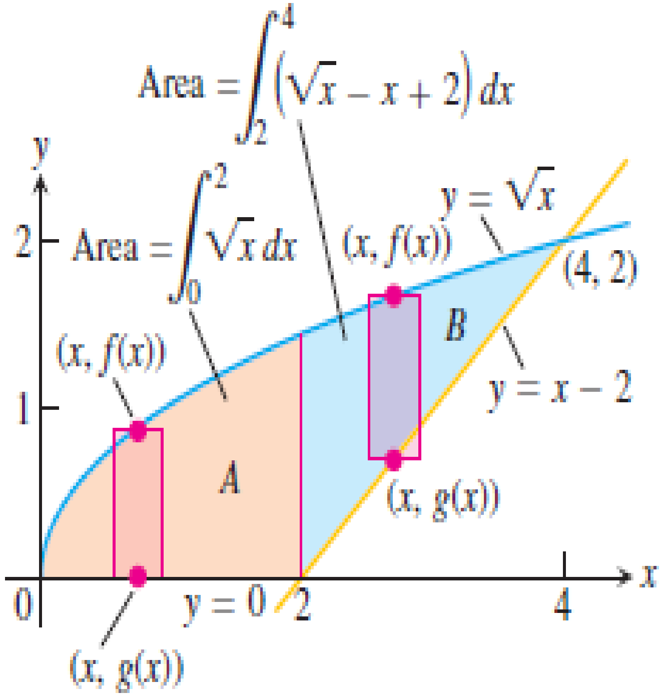
The area between the curves is

$$A = \int_a^b [f(x) - g(x)]\, dx = \int_{-1}^2 [(2 - x^2) - (-x)]\, dx$$
$$= \int_{-1}^2 (2 + x - x^2)\, dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^2$$
$$= \left(4 + \frac{4}{2} - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right) = \frac{9}{2}$$

EXAMPLE 10:

Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.

Solution:



We subdivide the region at into subregions *A and B, shown in Figure* we solve the equations

$\sqrt{x} = x - 2$	Equate $f(x)$ and $g(x)$.
$x = (x-2)^2 = x^2 - 4x + 4$	Square both sides.
$x^2 - 5x + 4 = 0$	Rewrite.
$(x-1)(x-4) = 0$	Factor.
$x = 1, \quad x = 4.$	Solve.

For $0 \le x \le 2$: $\quad f(x) - g(x) = \sqrt{x} - 0 = \sqrt{x}$

For $2 \le x \le 4$: $\quad f(x) - g(x) = \sqrt{x} - (x - 2) = \sqrt{x} - x + 2$

We add the area of subregions A and B to find the total area:

$$\text{Total area} = \underbrace{\int_0^2 \sqrt{x}\,dx}_{\text{area of } A} + \underbrace{\int_2^4 (\sqrt{x} - x + 2)\,dx}_{\text{area of } B}$$

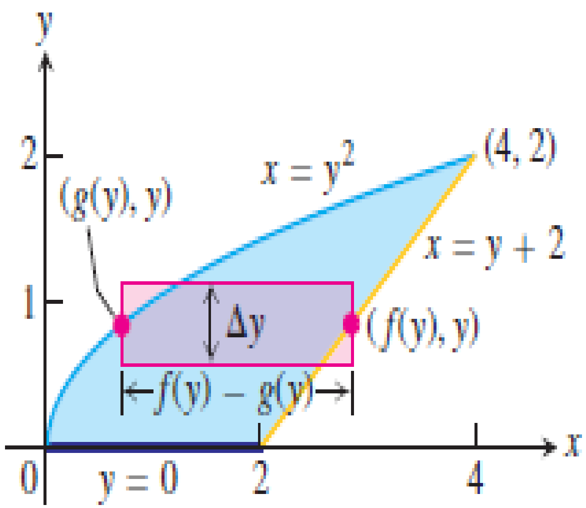
$$= \left[\frac{2}{3}x^{3/2}\right]_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4$$

$$= \frac{2}{3}(2)^{3/2} - 0 + \left(\frac{2}{3}(4)^{3/2} - 8 + 8\right) - \left(\frac{2}{3}(2)^{3/2} - 2 + 4\right)$$

$$= \frac{2}{3}(8) - 2 = \frac{10}{3}.$$

EXAMPLE 11:

Find the area of the region in Example 5 by integrating with respect to y.



Solution :The region's right-hand boundary is the line $x = y + 2$, so $f(y) = y + 2$.
The left-hand boundary is the curve $x = y^2$, so $g(y) = y^2$. The lower limit of integration is $y = 0$. We find the upper limit by solving

$$y + 2 = y^2 \quad \text{Equate } f(y) = y + 2 \text{ and } g(y) = y^2.$$

$$y^2 - y - 2 = 0 \quad \text{Rewrite.}$$

$$(y + 1)(y - 2) = 0 \quad \text{Factor.}$$

$$y = -1, \quad y = 2 \quad \text{Solve.}$$

The upper limit of integration is $b = 2$. (The value $y = -1$ gives a point of intersection *below* the x-axis.)

The area of the region is

$$A = \int_a^b [f(y) - g(y)]\,dy = \int_0^2 [y + 2 - y^2]\,dy$$

$$= \int_0^2 [2 + y - y^2]\,dy$$

$$= \left[2y + \frac{y^2}{2} - \frac{y^3}{3}\right]_0^2$$

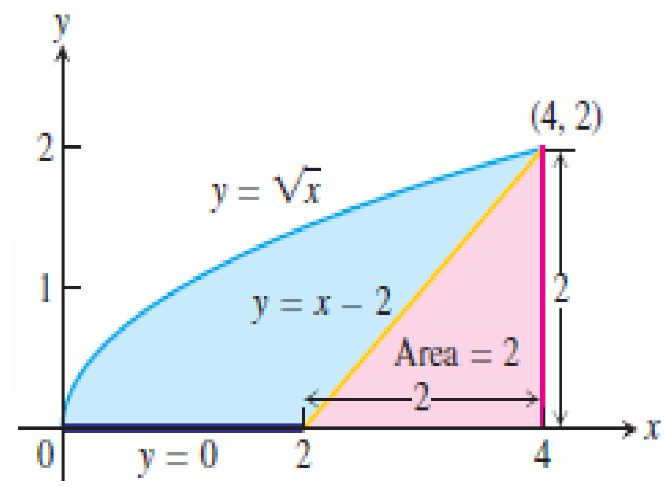
$$= 4 + \frac{4}{2} - \frac{8}{3} = \frac{10}{3}.$$

˝Combining Integrals with Formulas from Geometry

The way to find an area may be to combine calculus and geometry.

EXAMPLE 12: Find the area of the region, shown in figure,

Solution:



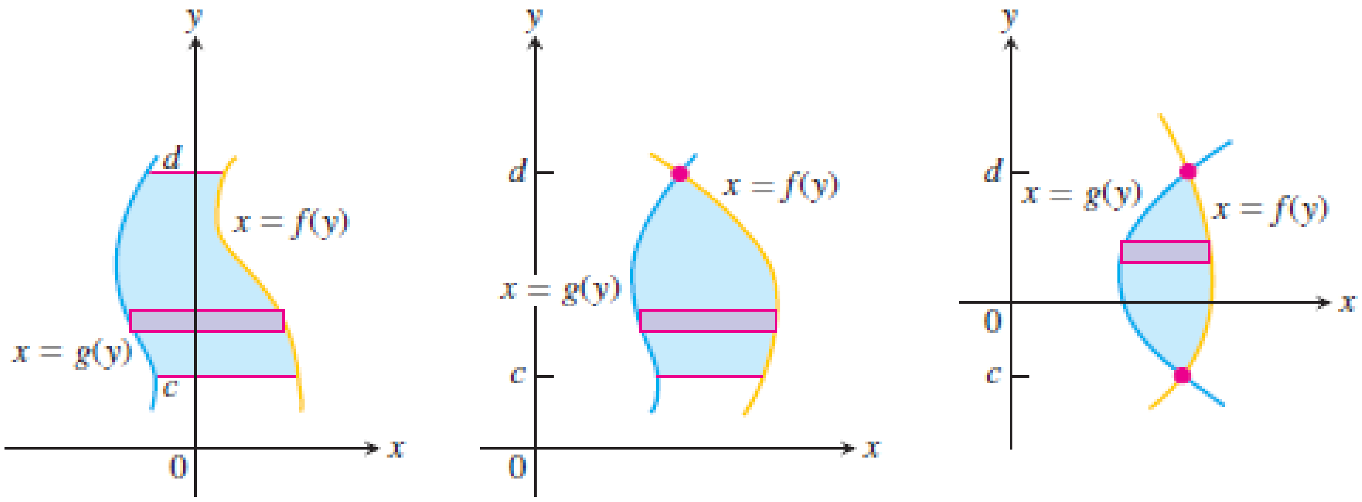
$$\text{Area} = \int_0^4 \sqrt{x}\, dx - \frac{1}{2}(2)(2)$$

$$= \frac{2}{3} x^{3/2} \Big]_0^4 - 2$$

$$= \frac{2}{3}(8) - 0 - 2 = \frac{10}{3}.$$

″Integration with Respect to *y*

If a region's bounding curves are described by functions of *y, the approximating rectangles* are horizontal instead of vertical and the basic formula has *y in place of x.* For regions like these.



use the formula

$$A = \int_c^d [f(y) - g(y)]\, dy.$$

In this equation f always denotes the right-hand curve and g the left-hand curve, so $f(y) - g(y)$ is nonnegative.

EXERCISES 8.4:

1. Use the Substitution Formula to evaluate the integrals in

2. a. $\int_0^1 r\sqrt{1-r^2}\,dr$ b. $\int_{-1}^1 r\sqrt{1-r^2}\,dr$

4. a. $\int_0^{\pi} 3\cos^2 x \sin x\,dx$ b. $\int_{2\pi}^{3\pi} 3\cos^2 x \sin x\,dx$

6. a. $\int_0^{\sqrt{7}} t(t^2+1)^{1/3}\,dt$ b. $\int_{-\sqrt{7}}^{0} t(t^2+1)^{1/3}\,dt$

9. a. $\int_0^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx$ b. $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx$

13. a. $\int_0^{2\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\,dz$ b. $\int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\,dz$

21. $\int_0^1 (4y-y^2+4y^3+1)^{-2/3}(12y^2-2y+4)\,dy$

2. Find the total areas of the shaded regions in Exercises

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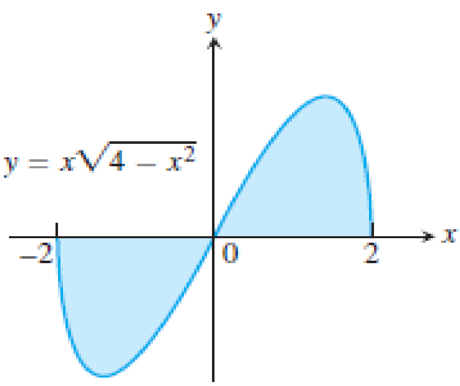

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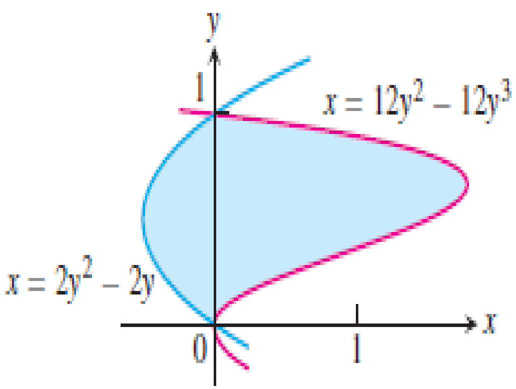

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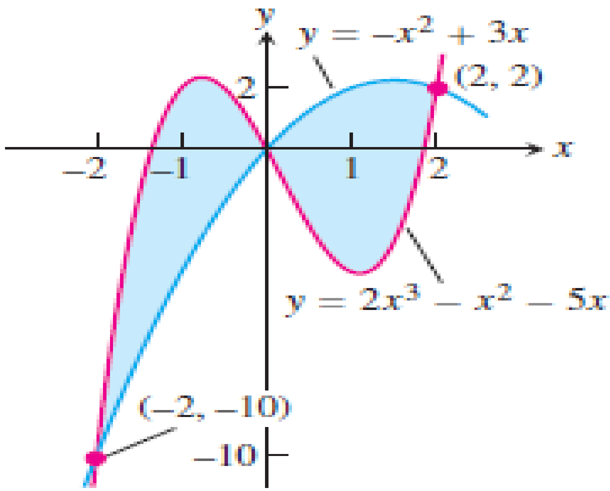

3. Find the areas of the regions enclosed by the lines and curves in Exercises .

45. $y = x^2$ and $y = -x^2 + 4x$

52. $x = y^2$ and $x = y + 2$

4. Find the areas of the regions enclosed by the curves in Exercises.

59. $4x^2 + y = 4$ and $x^4 - y = 1$

60. $x^3 - y = 0$ and $3x^2 - y = 4$