MATHEMATICS I

FIRST SEMESTER

Lec. 07

Applications of Derivatives

Outlines

- Extreme values of function (absolute maximum and minimum, local value).
- Critical Points
- Increasing and Decreasing Function
- Concave Up and Concave Down
- Points of Inflection
- Local Extrema
- Strategy for Graphing
- Newton-Raphson Method

- Absolute Maximum, Absolute Minimum

Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in } D$$

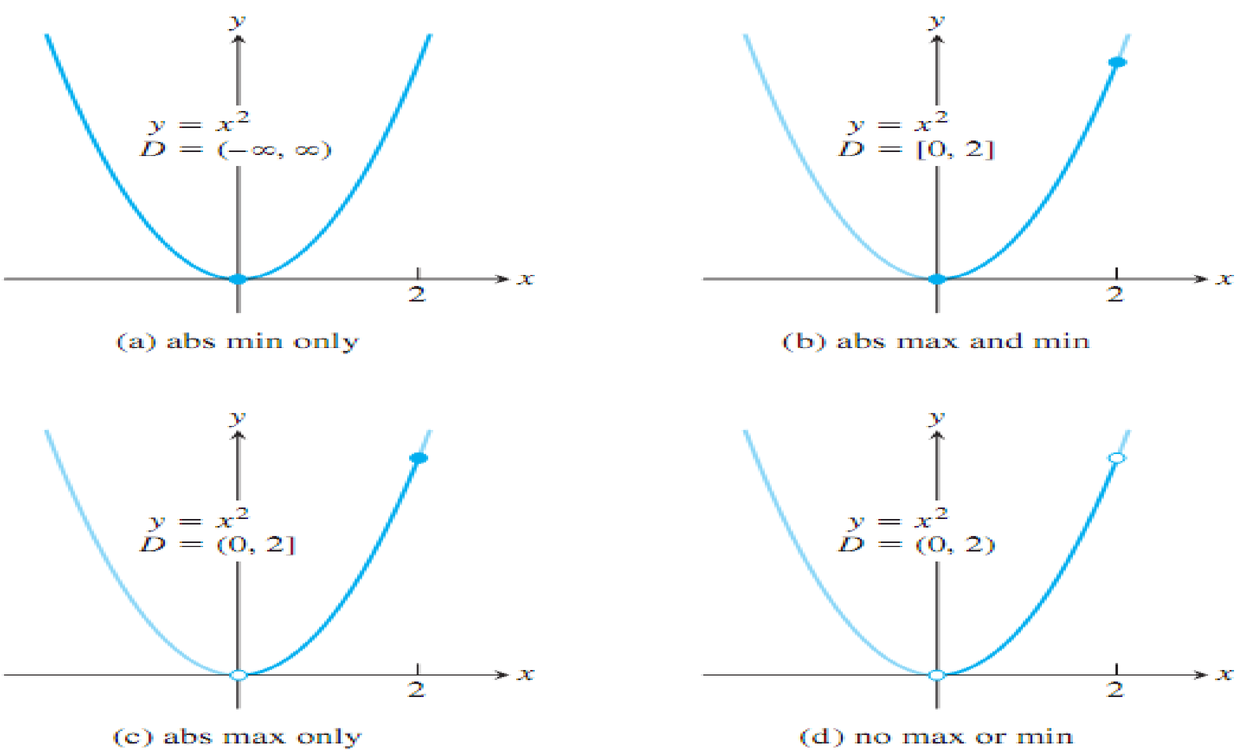
and an **absolute minimum** value on D at c if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in } D.$$

EXAMPLE 1 Exploring Absolute Extrema

The absolute extrema of the following functions on their domains can be seen in Figure

Function rule	Domain D	Absolute extrema on D
(a) $y = x^2$	$(-\infty, \infty)$	No absolute maximum. Absolute minimum of 0 at $x = 0$.
(b) $y = x^2$	$[0, 2]$	Absolute maximum of 4 at $x = 2$. Absolute minimum of 0 at $x = 0$.
(c) $y = x^2$	$(0, 2]$	Absolute maximum of 4 at $x = 2$. No absolute minimum.
(d) $y = x^2$	$(0, 2)$	No absolute extrema.



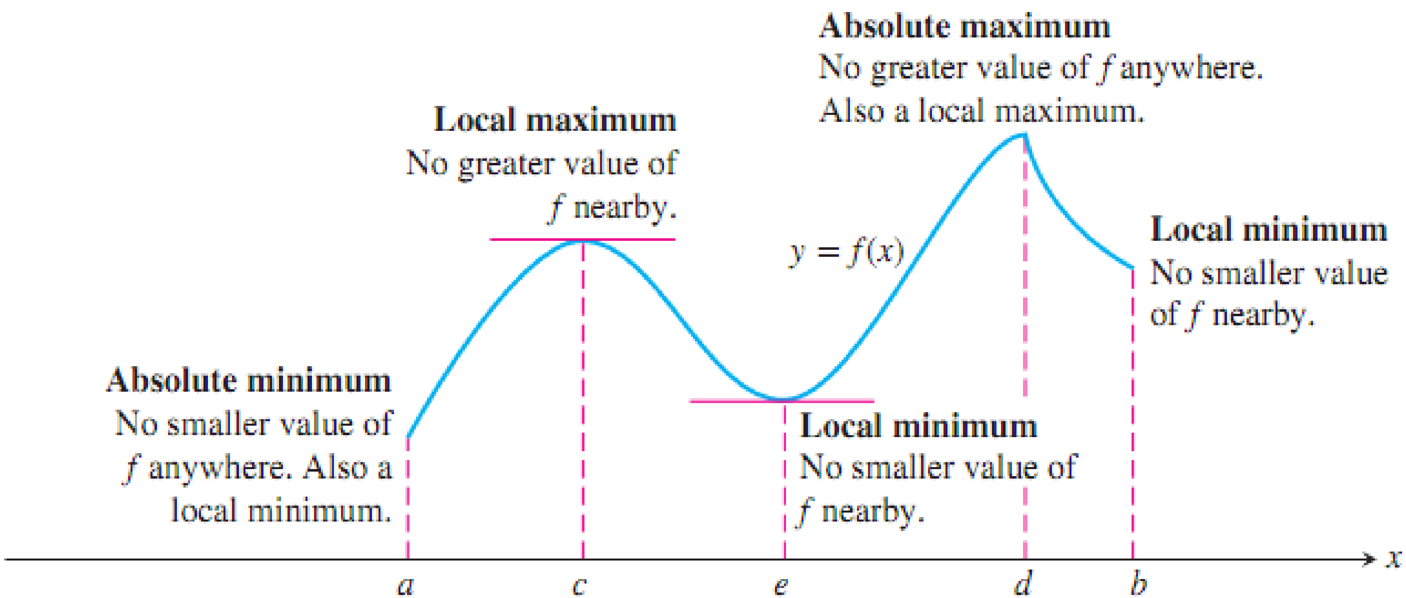
Local Maximum, Local Minimum

A function f has a **local maximum** value at an interior point c of its domain if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$

A function f has a **local minimum** value at an interior point c of its domain if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$



THEOREM The First Derivative Theorem for Local Extreme Values

If f has a local maximum or minimum value at an interior point c of its domain and if f' is defined at c, then

$$f'(c) = 0.$$

Critical Point

An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.

EXAMPLE 2 Find the absolute maximum and minimum values of $f(x) = x^2$ on $[-2, 1]$.

Solution The function is differentiable over its entire domain, so the only critical point is where $f'(x) = 2x = 0$, namely $x = 0$. We need to check the function's values at $x = 0$ and at the endpoints $x = -2$ and $x = 1$:

Critical point value: $f(0) = 0$

Endpoint values: $f(-2) = 4$

$f(1) = 1$

The function has an absolute maximum value of 4 at $x = -2$ and an absolute minimum value of 0 at $x = 0$.

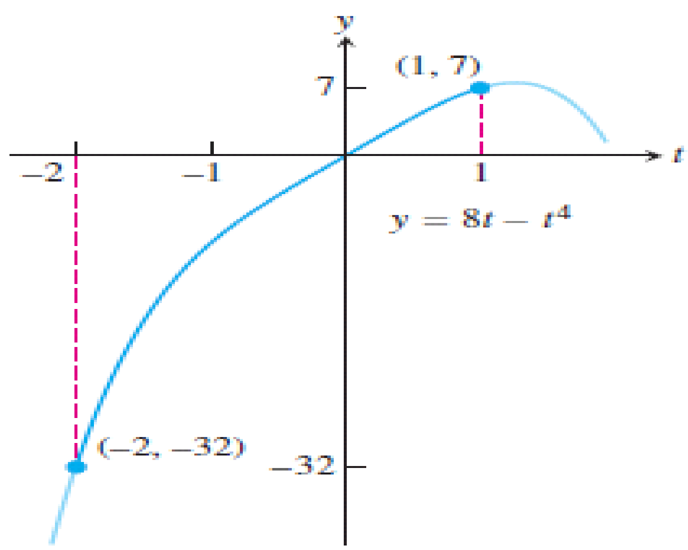
EXAMPLE 3 Absolute Extrema at Endpoints

Find the absolute extrema values of $g(t) = 8t - t^4$ on $[-2, 1]$.

Solution The function is differentiable on its entire domain, so the only critical points occur where $g'(t) = 0$. Solving this equation gives

$$8 - 4t^3 = 0 \qquad \text{or} \qquad t = \sqrt[3]{2} > 1,$$

a point not in the given domain. The function's absolute extrema therefore occur at the endpoints, $g(-2) = -32$ (absolute minimum), and $g(1) = 7$ (absolute maximum). See



EXAMPLE 4 Finding Absolute Extrema on a Closed Interval

Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-2, 3]$.

Solution We evaluate the function at the critical points and endpoints and take the largest and smallest of the resulting values.

The first derivative

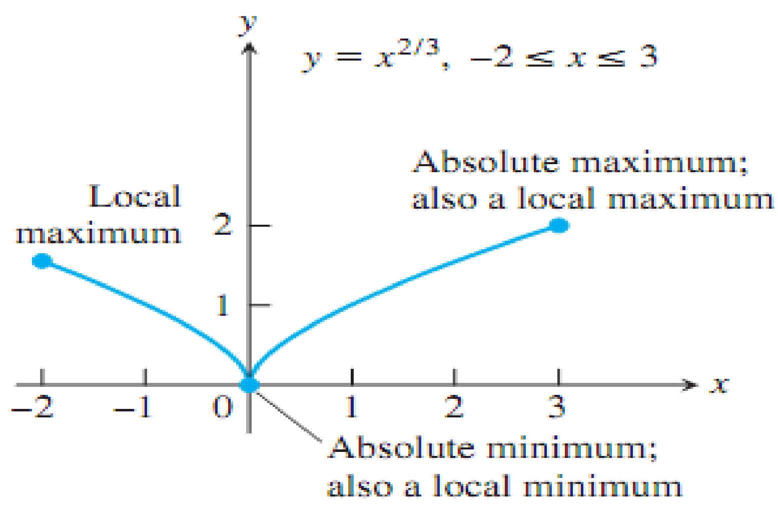
$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

has no zeros but is undefined at the interior point $x = 0$. The values of f at this one critical point and at the endpoints are

Critical point value: $f(0) = 0$

Endpoint values: $f(-2) = (-2)^{2/3} = \sqrt[3]{4}$

$f(3) = (3)^{2/3} = \sqrt[3]{9}$.



EXERCISES 7.1:

1. find the absolute maximum and minimum values of each function on the given interval. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

1. $f(x) = \frac{2}{3}x - 5, \quad -2 \le x \le 3$
2. $f(x) = -x - 4, \quad -4 \le x \le 1$
3. $f(x) = x^2 - 1, \quad -1 \le x \le 2$
4. $f(x) = 4 - x^2, \quad -3 \le x \le 1$

2. find the derivative at each critical point and determine the local extreme values.

1. $y = x^{2/3}(x + 2)$
2. $y = x\sqrt{4 - x^2}$
3. $y = x^{2/3}(x^2 - 4)$
4. $y = x^2\sqrt{3 - x}$

The Mean Value Theorem

To arrive at this theorem we first need Rolle's Theorem.

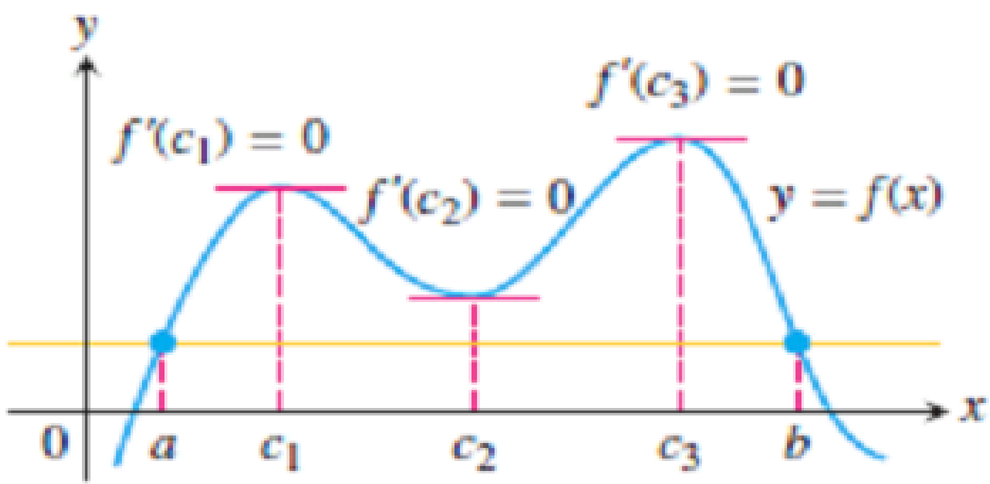
Rolle's Theorem:

Suppose that $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If

$$f(a) = f(b),$$

then there is at least one number c in (a, b) at which

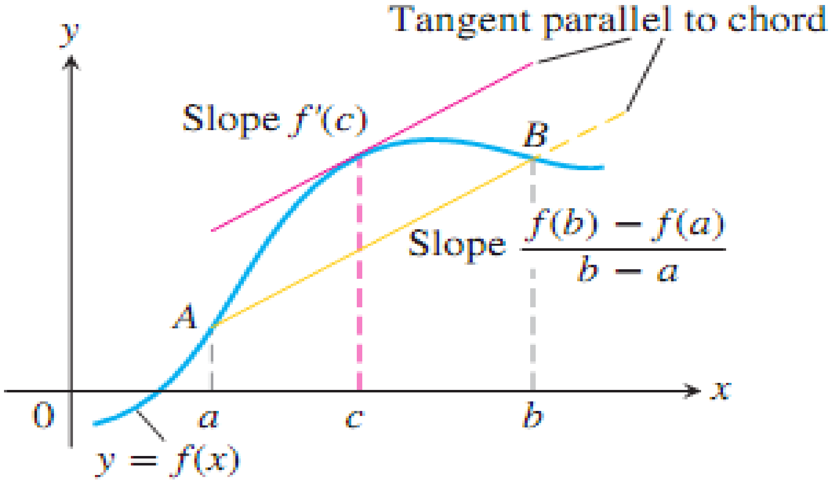
$$f'(c) = 0.$$



The Mean Value Theorem

Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

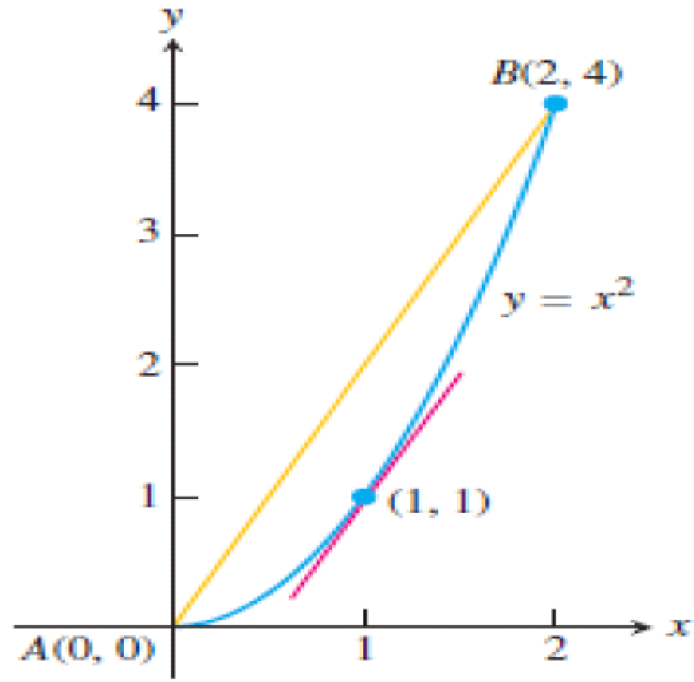


EXAMPLE 3 The function $f(x) = x^2$ (Figure 4.18) is continuous for $0 \le x \le 2$ and differentiable for $0 < x < 2$. Since $f(0) = 0$ and $f(2) = 4$, the Mean Value Theorem says that at some point c in the interval, the derivative $f'(x) = 2x$ must have the value $(4 - 0)/(2 - 0) = 2$. In this (exceptional) case we can identify c by solving the equation $2c = 2$ to get $c = 1$. ■

Solution: $$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$\frac{2^2-0}{2-0} = \frac{4-0}{2} = 2$$

$$f'(x)=2x \rightarrow f'(c) = 2c = 2 \rightarrow c = 1$$



EXERCISES **7.2:**

Finding *c* in the Mean Value Theorem

Find the value or values of *c* that satisfy the equation

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

in Exercises 1–4.

1. $f(x) = x^2 + 2x - 1, \quad [0, 1]$
2. $f(x) = x^{2/3}, \quad [0, 1]$
3. $f(x) = x + \frac{1}{x}, \quad \left[\frac{1}{2}, 2\right]$
4. $f(x) = \sqrt{x - 1}, \quad [1, 3]$

Monotonic Functions and The First Derivative Test

"Increasing Functions and Decreasing Functions:

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I.
2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I.

A function that is increasing or decreasing on I is called **monotonic** on I.

"First Derivative Test for Monotonic Functions:

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.

If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

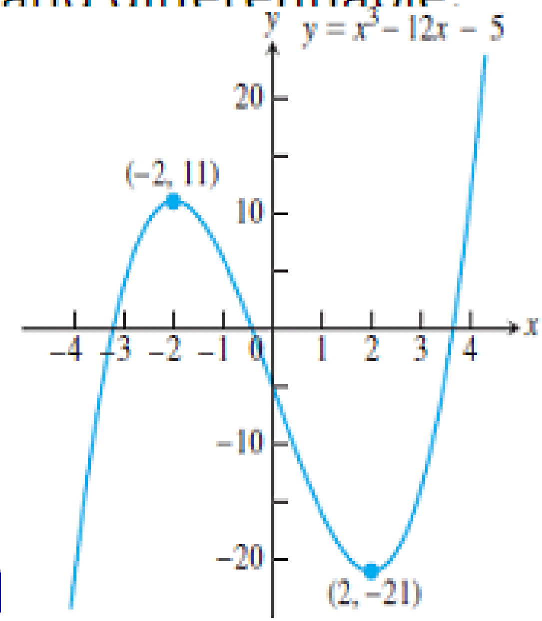
EXAMPLE 5 : Using the First Derivative Test for Monotonic Functions

Find the critical points of f(x) = $x^3 - 12x - 5$ and identify the intervals on which f is increasing and decreasing.

Solution: The function f is everywhere continuous and differentiable.

The first derivative,

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4)$$
$$= 3(x + 2)(x - 2)$$



is zero at x = -2 and x = 2.

These critical points subdivide the domain of f into intervals $(-\infty, -2)$,(-2,2),and$(2,\infty)$ and we determine the sign of f' by evaluating f at a point in each subinterval.The behavior of f is determined

Corollary 3 to each subinterval.

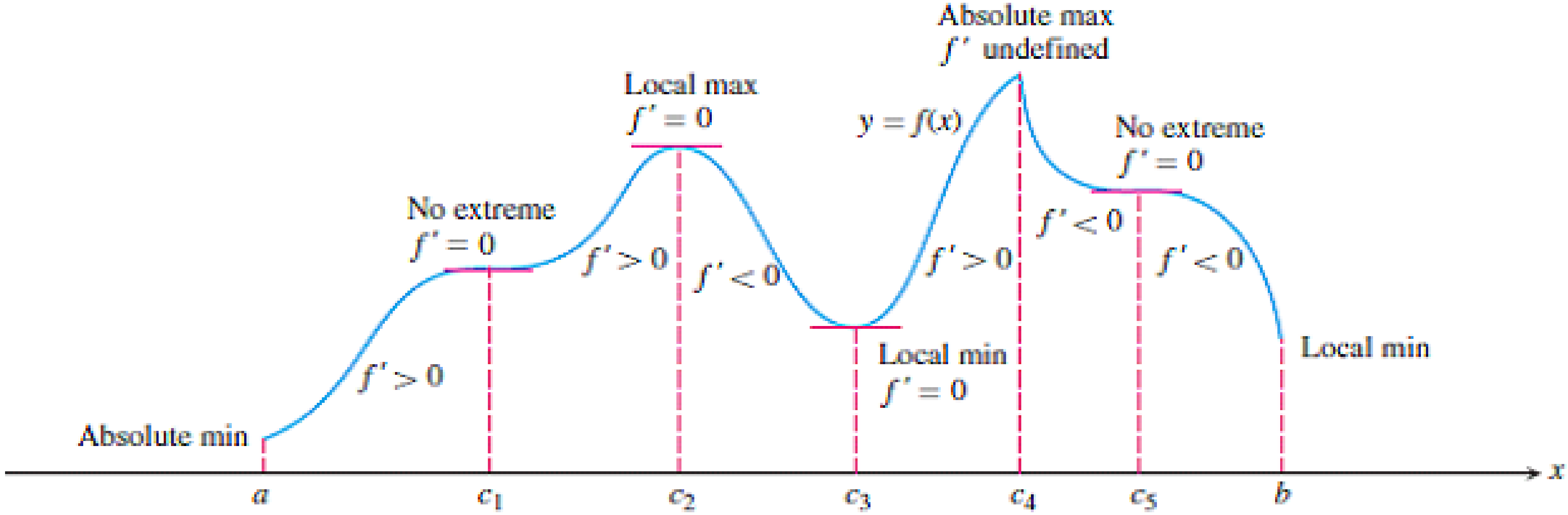
Intervals	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
f' Evaluated	$f'(-3) = 15$	$f'(0) = -12$	$f'(3) = 15$
Sign of f'	+	−	+
Behavior of f	increasing	decreasing	increasing

First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across c from left to right,

1. if f' changes from negative to positive at c, then f has a local minimum at c;
2. if f' changes from positive to negative at c, then f has a local maximum at c;
3. if f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no local extremum at c.

FIGURE , A function's first derivative tells how the graph rises and falls.



EXAMPLE 6: Using the First Derivative Test for Local Extrema,find the critical points of, $f(x) = x^{1/3}(x-4) = x^{4/3} - 4x^{1/3}$.

Identify the intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

Solution The function f is continuous at all x since it is the product of two continuous functions, $x^{1/3}$ and $(x-4)$. The first derivative

$$f'(x) = \frac{d}{dx}\left(x^{4/3} - 4x^{1/3}\right) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3}$$

$$= \frac{4}{3}x^{-2/3}\left(x-1\right) = \frac{4(x-1)}{3x^{2/3}}$$

is zero at $x = 1$ and undefined at $x = 0$. There are no endpoints in the domain, so the critical points $x = 0$ and $x = 1$ are the only places where f might have an extreme value.

The critical points partition the x-axis into intervals on which f' is either positive or negative. The sign pattern of f' reveals the behavior of f between and at the critical points. We can display the information in a table like the following:

Intervals	$x < 0$	$0 < x < 1$	$x > 1$
Sign of f'	$-$	$-$	$+$
Behavior of f	decreasing	decreasing	increasing

Corollary 3 to the Mean Value Theorem tells us that f decreases on $(-\infty, 0)$, decreases on $(0, 1)$, and increases on $(1, \infty)$. The First Derivative Test for Local Extrema tells us that f does not have an extreme value at $x = 0$ (f' does not change sign) and that f has a local minimum at $x = 1$ (f' changes from negative to positive).

The value of the local minimum is $f(1) = 1^{1/3}(1 - 4) = -3$. This is also an absolute minimum because the function's values fall toward it from the left and rise away from it on the right. Figure 4.24 shows this value in relation to the function's graph.

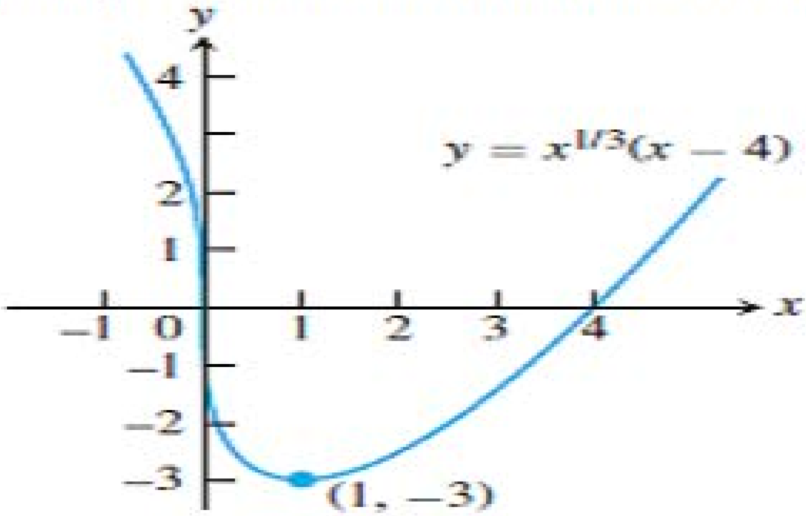


FIGURE 4.24 The function $f(x) = x^{1/3}(x - 4)$ decreases when $x < 1$ and increases when $x > 1$ (Example 2).

EXample7:

Given $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x + 8$, find (*a*) the critical points; (*b*) the intervals on which y is increasing and decreasing; and (*c*) the maximum and minimum values of y.

(*a*) $y' = x^2 + x - 6 = (x+3)(x-2)$. Setting $y' = 0$ gives the critical values $x = -3$ and 2. The critical points are $(-3, \frac{43}{2})$ and $(2, \frac{2}{3})$.

(*b*) When y' is positive, y increases; when y' is negative, y decreases.

When $x < -3$, say $x = -4$, $\quad y' = (-)(-) = +$, and y is increasing.

When $-3 < x < 2$, say $x = 0$, $\quad y' = (+)(-) = -$, and y is decreasing.

When $x > 2$, say $x = 3$, $\quad y' = (+)(+) = +$, and y is increasing.

These results are illustrated by the following diagram (see Fig. 13-3):

$x < -3$	$x = -3$	$-3 < x < 2$	$x = 2$	$x > 2$
$y' = +$ y increases		$y' = -$ y decreases		$y' = +$ y increases

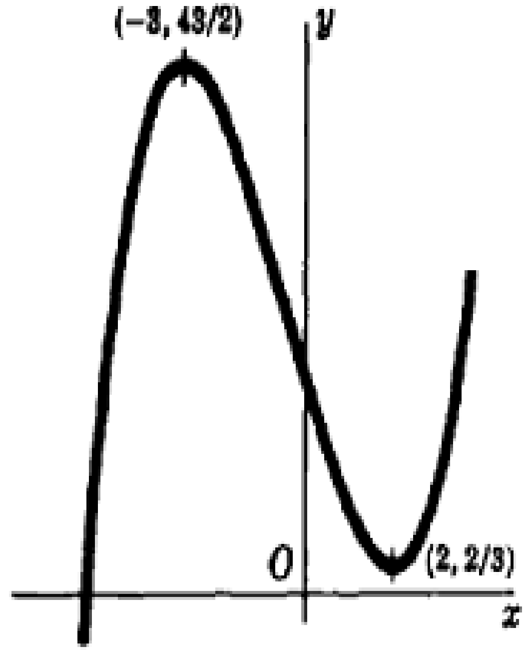


Fig. 13-3

(c) We test the critical values $x = -3$ and 2 for maxima and minima:

As x increases through -3, y' changes sign from + to −; hence at $x = -3$, y has a maximum value $\frac{43}{2}$.

As x increases through 2, y' changes sign from − to +; hence at $x = 2$, y has a minimum value $\frac{2}{3}$.

EXERCISES 7.3:

1. Answer the following questions about the functions whose derivatives are given in Exercises 1–5:

a. What are the critical points of f?

b. On what intervals is f increasing or decreasing?

c. At what points, if any, does f assume local maximum and minimum values?

1. $f'(x) = x(x - 1)$ 2. $f'(x) = (x - 1)^2(x + 2)$ 3. $f'(x) = (x - 1)^2(x + 2)^2$

4. $f'(x) = (x - 7)(x + 1)(x + 5)$ 5. $f'(x) = x^{-1/2}(x - 3)$

2. In Exercises 1–4:

a. Find the intervals on which the function is increasing and decreasing.

b. Then identify the function's local extreme values, if any, saying where they are taken on.

c. Which, if any, of the extreme values are absolute?

d. Support your findings with a graphing calculator or computer grapher.

1. $g(t) = -3t^2 + 9t + 5$

2. $h(x) = -x^3 + 2x^2$

3. $f(\theta) = 3\theta^2 - 4\theta^3$

4. $f(r) = 3r^3 + 16r$

Concavity and Curve Sketching

In this section we see how the second derivative gives information about the way the graph of a differentiable function bends or turns.

"Concavity

DEFINITION Concave Up, Concave Down

The graph of a differentiable function $y = f(x)$ is

(a) **concave up** on an open interval I if f' is increasing on I

(b) **concave down** on an open interval I if f' is decreasing on I.

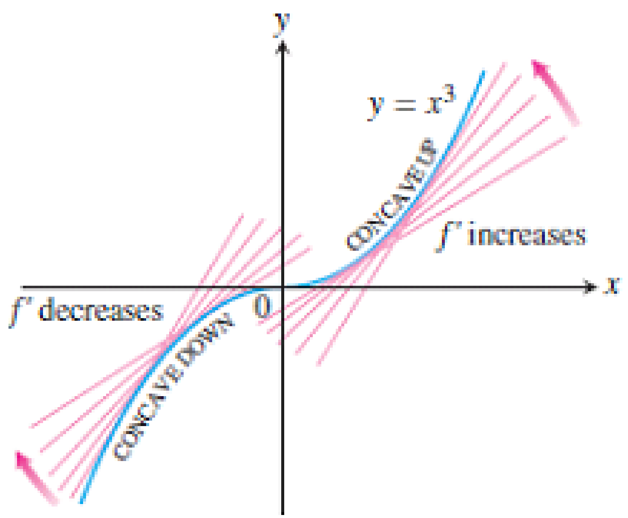


FIGURE 4.25 The graph of $f(x) = x^3$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$ (Example 1a).

"The Second Derivative Test for Concavity

Let $y = f(x)$ be twice-differentiable on an interval I.

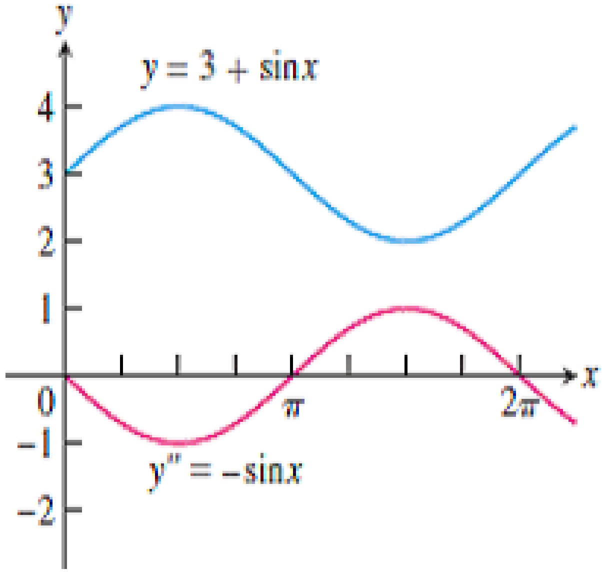
1. If $f'' > 0$ on I, the graph of f over I is concave up.
2. If $f'' < 0$ on I, the graph of f over I is concave down.

EXAMPLE 7: Determining Concavity, Determine the concavity of

$y = 3 + \sin x$ on $[0, 2\pi]$

Solution:

The graph of $y = 3 + \sin x$ is concave down on $(0, \pi)$, where $y'' = -\sin x$ is negative. It is concave up on $(\pi, 2\pi)$, where $y'' = -\sin x$ is Positive (see figuer).



Points of Inflection

The curve $y = 3 + \sin x$ in Example 2 changes concavity at the point $(\pi, 3)$. We call $(\pi, 3)$ a *point of inflection* of the curve.

DEFINITION **Point of Inflection**

A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

EXAMPLE 8: Using f′ and f″to Graph f Sketch a graph of the function $f(x) = x^4 - 4x^3 + 10$. using the following steps.

(a) Identify where the extrema of f occur.

(b) Find the intervals on which f is increasing and the intervals on which f is decreasing.

(c) Find where the graph of f is concave up and where it is concave down.

(d) Sketch the general shape of the graph for f.

(e) Plot some specific points, such as local maximum and minimum points, points of inflection,and intercepts. Then sketch the curve.

Solution f is continuous since $f'(x) = 4x^3 - 12x^2$ exists. The domain of f is $(-\infty, \infty)$, and the domain of f' is also $(-\infty, \infty)$. Thus, the critical points of f occur only at the zeros of f'. Since

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$$

the first derivative is zero at $x = 0$ and $x = 3$.

Intervals	$x < 0$	$0 < x < 3$	$3 < x$
Sign of f'	$-$	$-$	$+$
Behavior of f	decreasing	decreasing	increasing

(a) Using the First Derivative Test for local extrema and the table above, we see that there is no extremum at $x = 0$ and a local minimum at $x = 3$.

(b) Using the table above, we see that f is decreasing on $(-\infty, 0]$ and $[0, 3]$, and increasing on $[3, \infty)$.

(c) $f''(x) = 12x^2 - 24x = 12x(x - 2)$ is zero at $x = 0$ and $x = 2$.

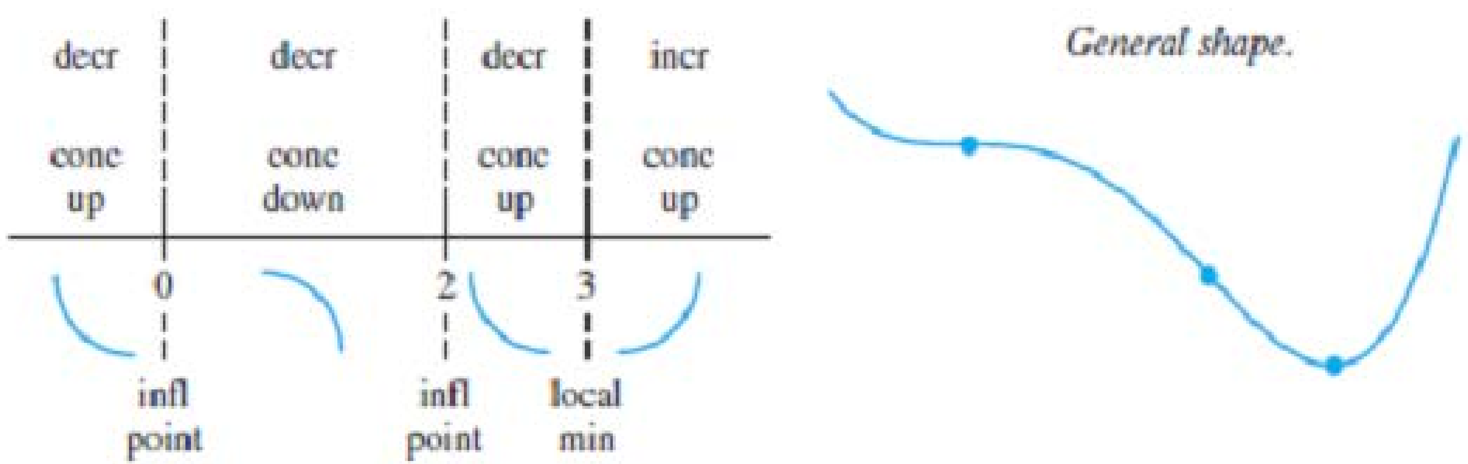
Intervals	$x < 0$	$0 < x < 2$	$2 < x$
Sign of f'	$+$	$-$	$+$
Behavior of f	concave up	concave down	concave up

We see that f is concave up on the intervals $(-\infty, 0)$ and $(2, \infty)$, and concave down on $(0, 2)$.

(d) Summarizing the information in the two tables above, we obtain

$x < 0$	$0 < x < 2$	$2 < x < 3$	$3 < x$
decreasing concave up	decreasing concave down	decreasing concave up	increasing concave up

The general shape of the curve is



(e) Plot the curve's intercepts (if possible) and the points where y' and y'' are zero. Indicate any local extreme values and inflection points. Use the general shape as a guide to sketch the curve. See figure.

(e) Plot the curve's intercepts (if possible) and the points where y' and y'' are zero. Indicate any local extreme values and inflection points. Use the general shape as a guide to sketch the curve. See figure.

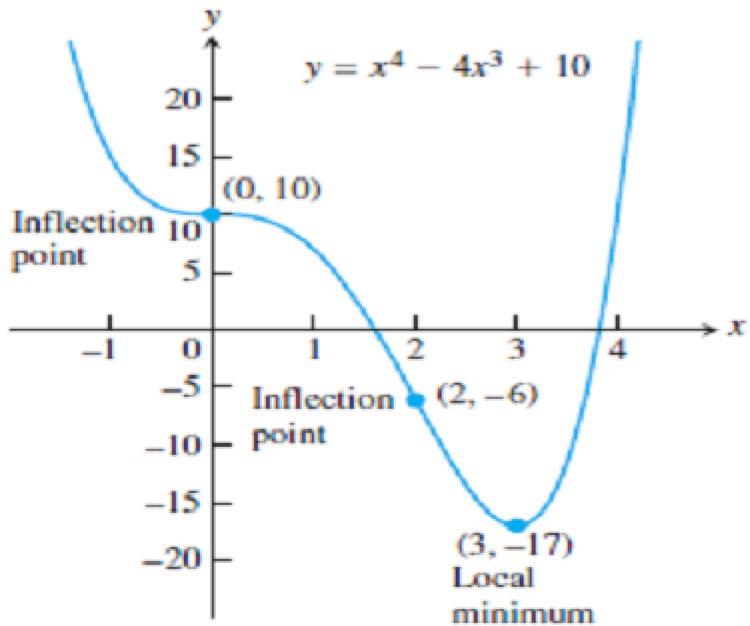


FIGURE 4. The graph of $f(x) = x^4 - 4x^3 + 10$

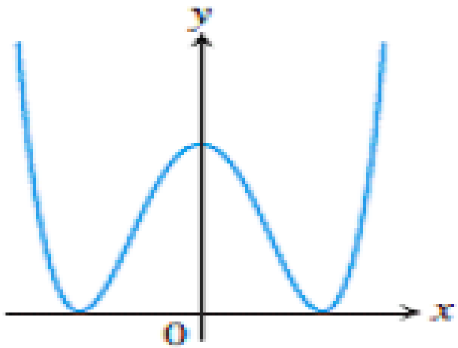
minimum

FIGURE 4. The graph of $f(x) = x^4 - 4x^3 + 10$

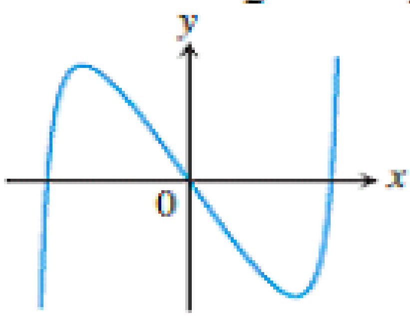
EXERCISES 7.4:

1. Identify the inflection points and local maxima and minima of the functions graphed in Exercises 1–3. Identify the intervals on which the functions are concave up and concave down.

1. $y = \frac{x^4}{4} - 2x^2 + 4$



2. $y = \tan x - 4x, -\frac{\pi}{2} < x < \frac{\pi}{2}$



2. Use the steps of the graphing procedure on page 272 to graph the equations in Exercises 1–5. Include the coordinates of any local extreme points and inflection points.

1. $y = (x-2)^3 + 1$

2. $y = -x^4 + 6x^2 - 4 = x^2(6 - x^2) - 4$

3. $y = 1 - 9x - 6x^2 - x^3$

4. $y = x^5 - 5x^4 = x^4(x - 5)$

5. $y = \frac{x^3}{3x^2 + 1}$

L'Hôpital's Rule

L'Hôpital's Rule (First Form)

Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist, and that $g'(a) \neq 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

EXAMPLE 1 Using L'Hôpital's Rule

(a) $\displaystyle \lim_{x \to 0} \frac{3x - \sin x}{x} = \left.\frac{3 - \cos x}{1}\right|_{x=0} = 2$

(b) $\displaystyle \lim_{x \to 0} \frac{\sqrt{1 + x} - 1}{x} = \left.\frac{\frac{1}{2\sqrt{1 + x}}}{1}\right|_{x=0} = \frac{1}{2}$

L'Hôpital's Rule (Stronger Form)

Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side exists.

EXAMPLE 2 Applying the Stronger Form of L'Hôpital's Rule

(a) $\displaystyle \lim_{x \to 0} \frac{\sqrt{1+x} - 1 - x/2}{x^2}$ $\quad \frac{0}{0}$

$$= \lim_{x \to 0} \frac{(1/2)(1+x)^{-1/2} - 1/2}{2x} \qquad \text{Still } \tfrac{0}{0}\text{; differentiate again.}$$

$$= \lim_{x \to 0} \frac{-(1/4)(1+x)^{-3/2}}{2} = -\frac{1}{8} \qquad \text{Not } \tfrac{0}{0}\text{; limit is found.}$$

(b) $\displaystyle \lim_{x \to 0} \frac{x - \sin x}{x^3}$ $\quad \frac{0}{0}$

$$= \lim_{x \to 0} \frac{1 - \cos x}{3x^2} \qquad \text{Still } \tfrac{0}{0}$$

$$= \lim_{x \to 0} \frac{\sin x}{6x} \qquad \text{Still } \tfrac{0}{0}$$

$$= \lim_{x \to 0} \frac{\cos x}{6} = \frac{1}{6} \qquad \text{Not } \tfrac{0}{0}\text{; limit is found.}$$

EXERCISES 7.5:

1. In Exercises 1–6, use l'Hôpital's Rule to evaluate the limit.

1. $\lim_{x \to 0} \frac{\sin 5x}{x}$ 2. $\lim_{x \to 1} \frac{x^3 - 1}{4x^3 - x - 3}$ 3. $\lim_{x \to \infty} \frac{2x^2 + 3x}{x^3 + x + 1}$

2. Use l'Hôpital's Rule to find the limits in Exercises 1–3.

1. $\lim_{x \to \pi/2} \frac{2x - \pi}{\cos x}$ 2. $\lim_{h \to 0} \frac{\sin(a + h) - \sin a}{h}$ 3. $\lim_{x \to 2} \frac{\sqrt{x^2 + 5} - 3}{x^2 - 4}$

Newton-Raphson method to find the roots of nonlinear algebraic equation

In this section we study a numerical method, called *Newton's method* or the *Newton–Raphson method*, which is a technique to approximate the solution to an equation f(x) = 0. Essentially it uses tangent lines in place of the graph of y = f(x) near the points where f is zero. (A value of x where f is zero is a *root* of the function f and a *solution* of the equation f(x) = 0.)

We can derive a formula for generating the successive approximations in the following way. Given the approximation x_n, the point-slope equation for the tangent to the curve at $(x_n, f(x_n))$ is

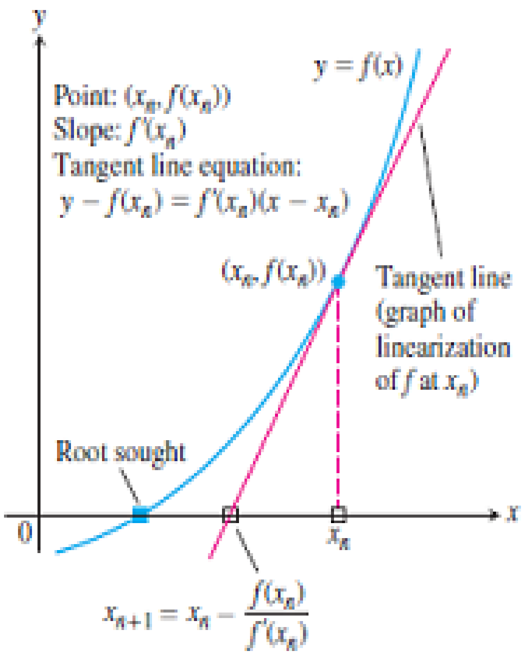
$$y = f(x_n) + f'(x_n)(x - x_n).$$

We can find where it crosses the x-axis by setting $y = 0$ (Figure 4.44).

$$0 = f(x_n) + f'(x_n)(x - x_n)$$

$$-\frac{f(x_n)}{f'(x_n)} = x - x_n$$

$$x = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \text{If } f'(x_n) \neq 0$$



Procedure for Newton's Method

1. Guess a first approximation to a solution of the equation $f(x) = 0$. A graph of $y = f(x)$ may help.
2. Use the first approximation to get a second, the second to get a third, and so on, using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad \text{if } f'(x_n) \neq 0 \tag{1}$$

EXAMPLE 1: **Applying Newton's Method**

Find the positive root of the equation

$$f(x) = x^2 - 2 = 0.$$

Solution With $f(x) = x^2 - 2$ and $f'(x) = 2x$, Equation (1) becomes

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n}$$

	Error	Number of correct digits
$x_0 = 1$	−0.41421	1
$x_1 = 1.5$	0.08579	1
$x_2 = 1.41667$	0.00246	3
$x_3 = 1.41422$	0.00001	5

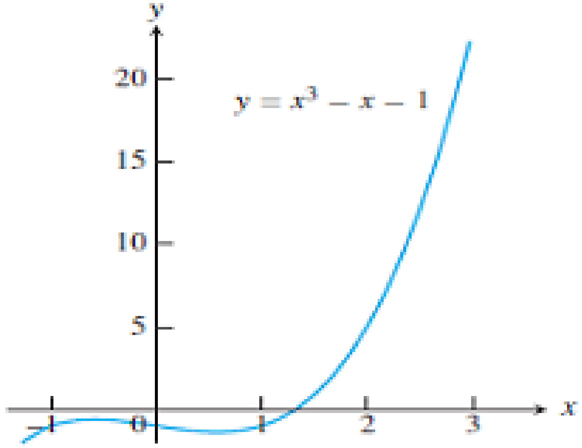
EXAMPLE 13: Using Newton's Method

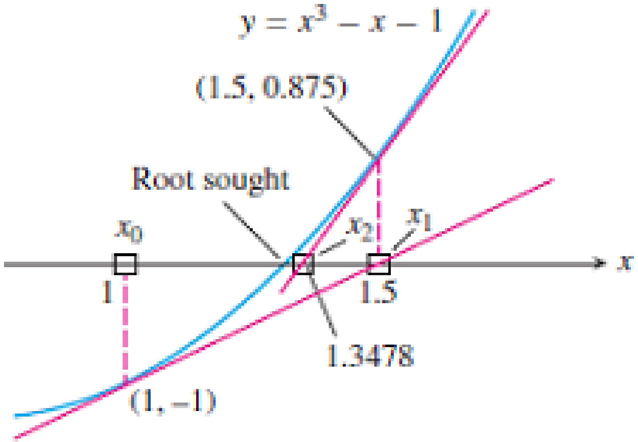
Find the *x*-coordinate of the point where the curve $y = x^3 - x$ crosses the horizontal line $y = 1$.

Solution The curve crosses the line when $x^3 - x = 1$ or $x^3 - x - 1 = 0$. When does $f(x) = x^3 - x - 1$ equal zero? Since $f(1) = -1$ and $f(2) = 5$, we know by the Intermediate Value Theorem there is a root in the interval $(1, 2)$ (Figure 4.45).

We apply Newton's method to f with the starting value $x_0 = 1$. The results are displayed in Table 4.1 and Figure 4.46.

At $n = 5$, we come to the result $x_6 = x_5 = 1.3247\,17957$. When $x_{n+1} = x_n$, Equation (1) shows that $f(x_n) = 0$. We have found a solution of $f(x) = 0$ to nine decimals. ■





In figure we have indicated that the process in Example 2 might have started at the point $B_0(3, 23)$ on the curve, with $x_0 = 3$. Point B_0 is quite far from the x-axis, but the Tangent at B_0 crosses the x-axis at about (2.12, 0), So x_1 is still an improvement over x_0 . If we use Equation (1) repeatedly as before, with $f(x) = x^3 - x - 1$ and $f'(x) = 3x^2 - 1$, we confirm the nine-place solution $x_7 = x_6 = 1.3247\ 17957$ in seven steps.

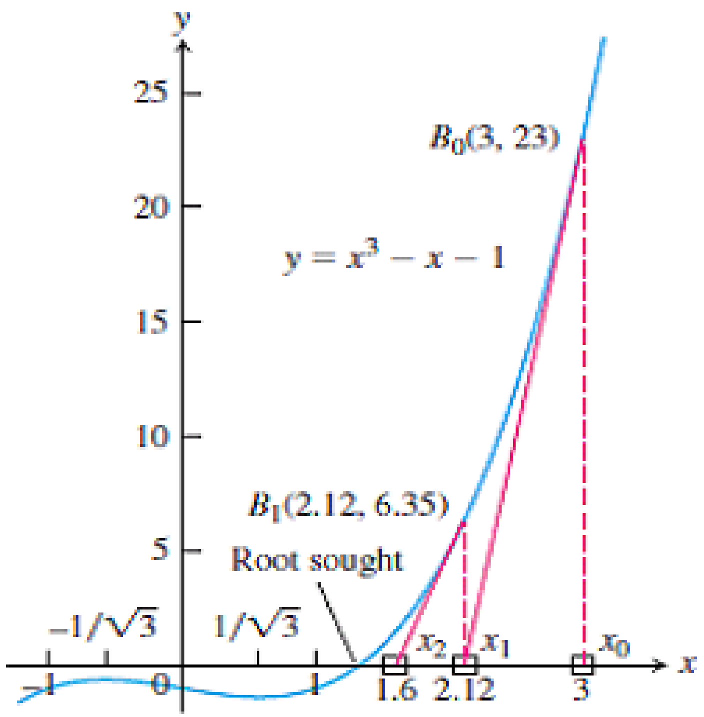


TABLE 4.1 The result of applying Newton's method to $f(x) = x^3 - x - 1$ with $x_0 = 1$

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	1	−1	2	1.5
1	1.5	0.875	5.75	1.3478 26087
2	1.3478 26087	0.1006 82173	4.4499 05482	1.3252 00399
3	1.3252 00399	0.0020 58362	4.2684 68292	1.3247 18174
4	1.3247 18174	0.0000 00924	4.2646 34722	1.3247 17957
5	1.3247 17957	−1.8672E-13	4.2646 32999	1.3247 17957

EXAMPLE 14:

Use Newton's method to estimate the one real solution $x^3 + 3x + 1 = 0$. of Start with $x_0 = 0$ and then find x_2 .

solution:

$$f(x) = x^3 + 3x + 1 \rightarrow f'(x) = 3x^2 + 3$$

When $x_{0=0}$, so $x_{1=} x_n - \frac{f(x_n)}{f'(x_n)}$ → $x_{1=} 0 - \frac{1}{3}$

then $x_2 = -\frac{1}{3} - \frac{-(\frac{1}{3})^3 + 3\left(-\frac{1}{3}\right) + 1}{3(-\frac{1}{3})^2 + 3} = -\frac{29}{90} = -0.32222$

EXERCISES 7.6:

1. Use Newton's method to estimate the solutions of the equation $x^2 + x - 1 = 0$. Start with $x_0 = -1$ for the left-hand solution and with $x_0 = 1$ for the solution on the right. Then, in each case, find x_2.
2. Use Newton's method to estimate the one real solution of $x^3 + 3x + 1 = 0$. Start with $x_0 = 0$ and then find x_2.
3. Use Newton's method to estimate the two zeros of the function $f(x) = x^4 + x - 3$. Start with $x_0 = -1$ for the left-hand zero and with $x_0 = 1$ for the zero on the right. Then, in each case, find x_2.
4. Use Newton's method to estimate the two zeros of the function $f(x) = 2x - x^2 + 1$. Start with $x_0 = 0$ for the left-hand zero and with $x_0 = 2$ for the zero on the right. Then, in each case, find x_2.
5. Use Newton's method to find the positive fourth root of 2 by solving the equation $x^4 - 2 = 0$. Start with $x_0 = 1$ and find x_2.
6. Use Newton's method to find the negative fourth root of 2 by solving the equation $x^4 - 2 = 0$. Start with $x_0 = -1$ and find x_2.

7. Use Newton's method to find the two negative zeros of $f(x) = x^3 - 3x - 1$ to five decimal places.

8. **Real solutions of a quartic** Use Newton's method to find the two real solutions of the equation $x^4 - 2x^3 - x^2 - 2x + 2 = 0$.

9. Solve the equation $x = 1/(x^2 + 1)$ with Newton's method.