MATHEMATICS I

FIRST SEMESTER

Lec. 02

Functions and Their Graphs

Outlines

- Function
- Domain and Range
- Graphs of Functions
- Piecewise-Defined Functions
- Types of functions
- Even Functions and Odd Functions: Symmetry
- Composite Functions
- Shifting a Graph of a Function

  -Vertical Shifts

  - Horizontal Shifts

- **Functions**

A function from a set D to a set Y is a rule that assigns a *unique* (single) element $f(x) \epsilon Y$ to each element $x \epsilon D$.

"A symbolic way to say "y is a function of x" is by writing

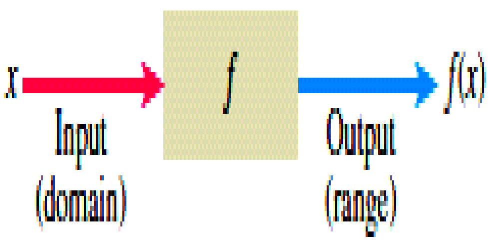
$y = f(x)$ ("y equals f of x").

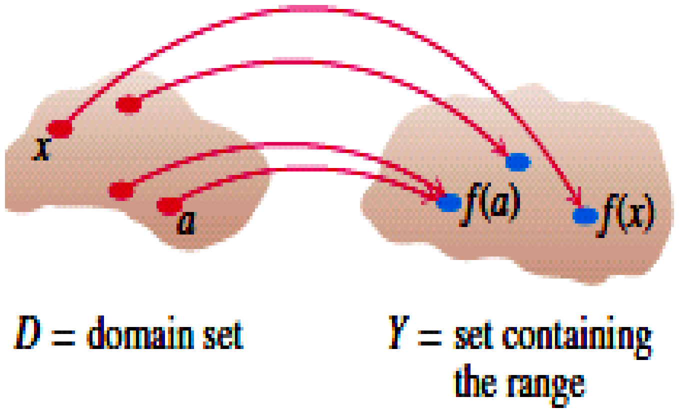
The symbol f represents the function. The letter x, called the **independent variable**, represents the input value of f, and y, the **dependent variable**, represents the corresponding output **value** of f at x.

• **Domain and Range**

Domain: The set from which the value of x can be chosen is called the **domain** of the function.

Range: The set of all values of y is called the **range** of the function.





Example 1:

1. Given $f(x) = \dfrac{x-1}{x^2+2}$, find (*a*) $f(0)$; (*b*) $f(-1)$; (*c*) $f(2a)$; (*d*) $f(1/x)$; (*e*) $f(x+h)$.

(*a*) $f(0) = \dfrac{0-1}{0+2} = -\dfrac{1}{2}$ (*b*) $f(-1) = \dfrac{-1-1}{1+2} = -\dfrac{2}{3}$ (*c*) $f(2a) = \dfrac{2a-1}{4a^2+2}$

(*d*) $f(1/x) = \dfrac{1/x-1}{1/x^2+2} = \dfrac{x-x^2}{1+2x^2}$ (*e*) $f(x+h) = \dfrac{x+h-1}{(x+h)^2+2} = \dfrac{x+h-1}{x^2+2hx+h^2+2}$

2. Determine the domains of the functions

(*a*) $y = \sqrt{4-x^2}$; (*b*) $y = \sqrt{x^2-16}$; (*c*) $y = \dfrac{1}{x-2}$;

(*d*) $y = \dfrac{1}{x^2-9}$; (*e*) $y = \dfrac{x}{x^2+4}$.

(*a*) Since y must be real, $4-x^2 \geq 0$, or $x^2 \leq 4$. The domain is the interval $-2 \leq x \leq 2$.
(*b*) Here, $x^2-16 \geq 0$, or $x^2 \geq 16$. The domain consists of the intervals $x \leq -4$ and $x \geq 4$.
(*c*) The function is defined for every value of x except 2.
(*d*) The function is defined for $x \neq \pm 3$.
(*e*) Since $x^2+4 \neq 0$ for all x, the domain is the set of all real numbers.

EXAMPLE 2

Determine the domains and ranges of these functions.

$y = x^2$
$y = 1/x$
$y = \sqrt{x}$
$y = \sqrt{4 - x}$
$y = \sqrt{1 - x^2}$

$\rightarrow$

Function	Domain (x)	Range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

Solution: "The formula y =x^2 gives a real *y*-value for any real number *x*, so the domain is $(-\infty,\infty)$ The range is$[0,\infty)$ because the square of any real number is nonnegative .

"The formula y = $\frac{1}{x}$ gives a real *y*-value for every *x* except x = 0, *We cannot divide any number by zero*. so the domain and range same are $(-\infty, 0) \cup (0,\infty)$.

" The formula y = $\sqrt{x}$ gives a real *y*-value only if $x \geq 0$, so the domain and range same are $[0, \infty)$.

"in y = $\sqrt{(4-x}$, the quantity 4 – x cannot be negative. That is, 4 – x ≥ 0, or x ≤ 4.
so the domain is (–∞,4]. The range is [0, ∞), the set of all nonnegative numbers.

" The formula y = $\sqrt{1-x^2}$ gives a real *y*-value for every *x* in the closed interval
From −1 to 1. so the domain is [–1 , 1]. the range of is [0, 1].

EXERCISES 2.1: find the domain and range of each function.

1. $f(x) = 1 + x^2$
2. $f(x) = 1 - \sqrt{x}$
3. $F(t) = \dfrac{1}{\sqrt{t}}$
4. $F(t) = \dfrac{1}{1 + \sqrt{t}}$
5. $g(z) = \sqrt{4 - z^2}$
6. $g(z) = \dfrac{1}{\sqrt{4 - z^2}}$

• Graphs of Functions

If f is a function with domain D, its **graph** consists of the points in the Cartesian plane whose coordinates are the input-output pairs for f. In set notation, the graph is $\{(x, f(x)) \mid x \in D\}$. To graph the function we need,

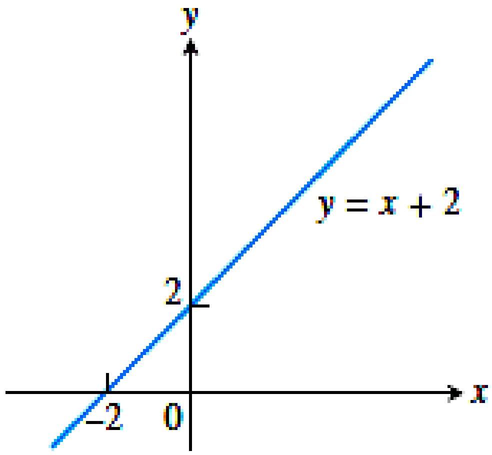
1. Make a table of *xy-pairs that satisfy the function rule.*

2. Plot the points (x, y).

3. Draw a smooth curve through the plotted points. Label the curve with its equation.

Example 3: Sketching a Graph

1. Graph the function $f(x) = x + 2$



Solution: The graph of the function $f(x) = x + 2$ is the set of points with coordinates (x, y) for which $y = x + 2$. Its graph is sketched in Figure.

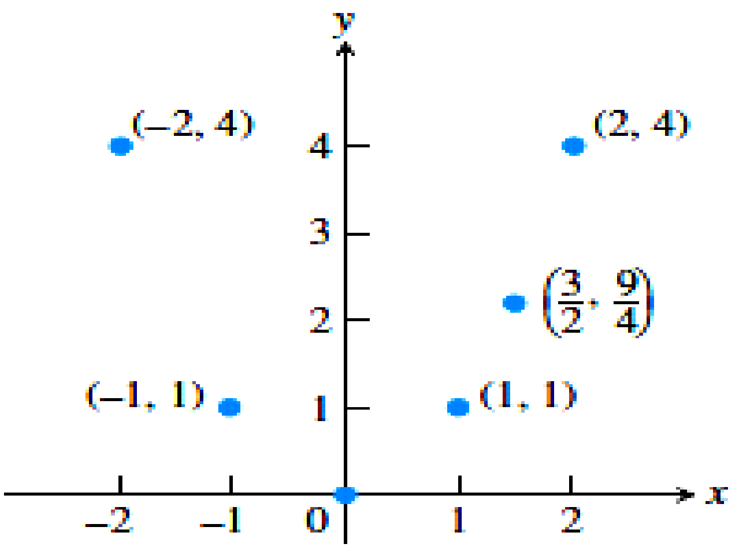
2. Graph the function Y = x^2 over the interval [–2, 2].

Solution:

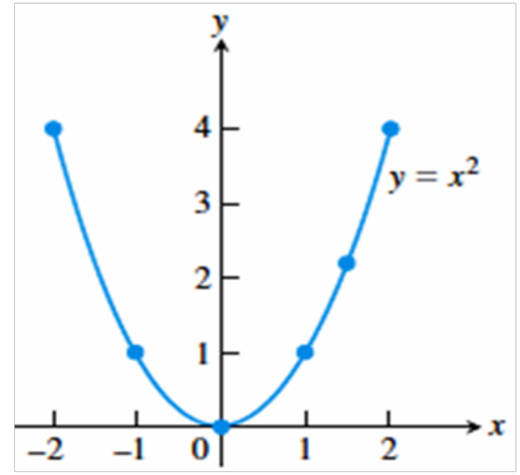
1. Make a table of *xy*-pairs that satisfy the function rule,

x	$y = x^2$
-2	4
-1	1
0	0
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4

2. Plot the points (*x*, *y*) whose coordinates appear in the table.



3. Draw a smooth curve through the plotted points.

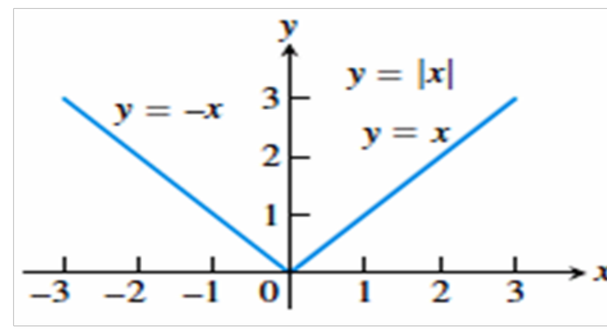


•Piecewise-Defined Functions

Sometimes a function is described by using different formulas on different parts of its domain. One example is the **absolute value function**

$$|X| = \begin{cases} X, & X \geq 0 \\ -X, & X < 0, \end{cases}$$

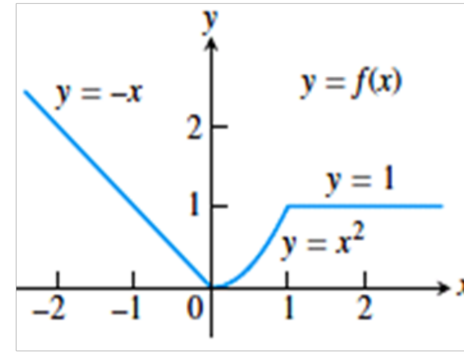
whose graph is given in Figure.



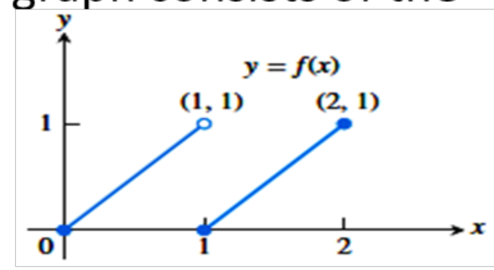
EXAMPLE 4 Graphing Piecewise-Defined Functions, The function

$$F(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ 1, & x > 0 \end{cases}$$

The values of f are given by: $y = -x$ when $x < 0$, $y = x^2$ when $0 \le x \le 1$, and $y = 1$ when $x > 1$. Figure shows the graph.



EXAMPLE 5: Write a formula for the function y = *f(x)* whose graph consists of the two line segments in Figure shows the graph.



Solution: We find formula for the segment from (0, 0) to (1, 1) The line through (0, 0) and (1, 1) has slope

$m = \frac{\Delta y}{\Delta x} = \frac{(1-0)}{(1-0)} = 1$, and *y*-intercept $b = 0$. Its slope-intercept equation is $y = x$.

"The Segment from (1, 0) to (2, 1) The line through (1, 0) and (2, 1) has slope

$m = \frac{\Delta y}{\Delta x} = \frac{(1-0)}{(2-1)} = 1$, and passes through the point (1, 0). The point slope equation for the line is $y = 0 + 1(x - 1)$, or $y = x - 1$.

Piecewise formula Combining the formulas for the two pieces of the graph, obtain

$$F(x) = \begin{cases} x, & 0 \le x < 1 \\ x - 1 & 1 \le x \le 2 \end{cases}$$

..

EXERCISES 2.2

1) Find the domain and graph of each functions.

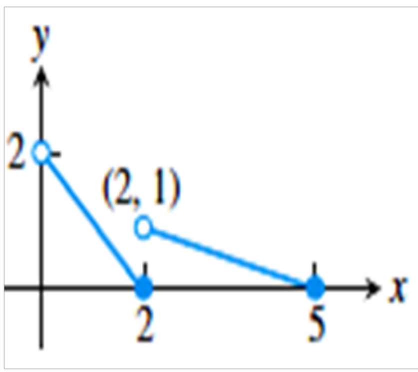
1. $f(x) = 1 - 2x - x^2$ 2. $g(x) = \sqrt{-x}$ 3. $F(t) = \frac{t}{|t|}$ 4. $g(x) = \sqrt{|x|}$

2) Graph the following functions.

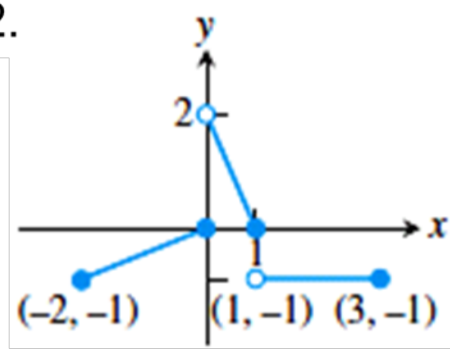
1. $g(x) = \begin{cases} 1 - x, & 0 \leq x \leq 1 \\ 2 - x, & 1 < x \leq 2 \end{cases}$ 2. $G(x) = \begin{cases} 1/x, & x < 0 \\ x, & 0 < x \end{cases}$

3) Find a formula for each function graphed.

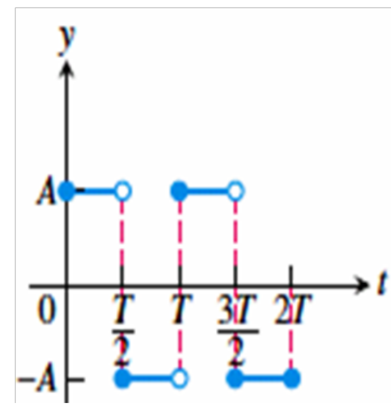
1.



2.

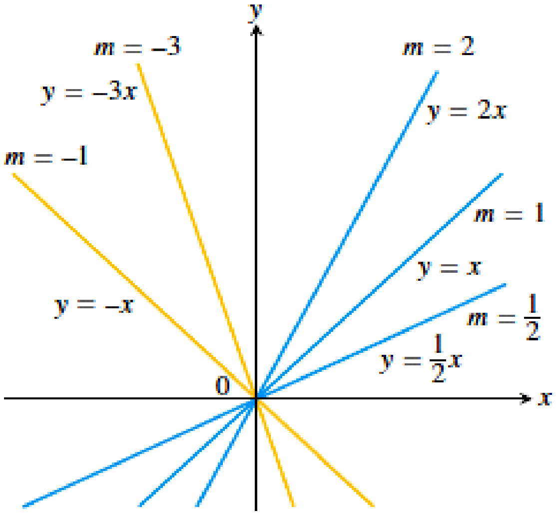


3.

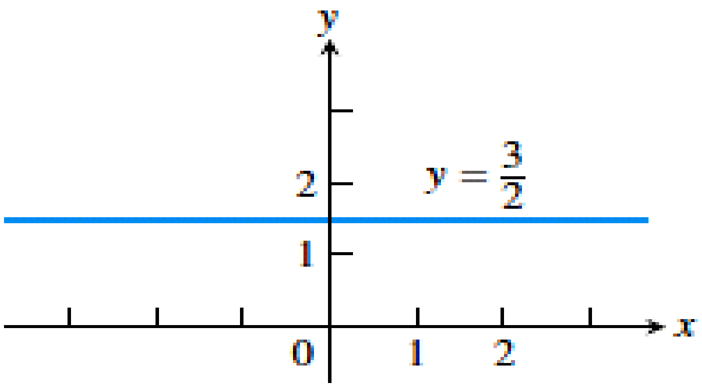


•Types of functions

1. Linear Functions: A function of the form $f(x) = mx + b$ for constants m and b, is called a **linear function**. Figure shows an array of lines $f(x) = mx$ where b = 0, so these lines pass through the origin.



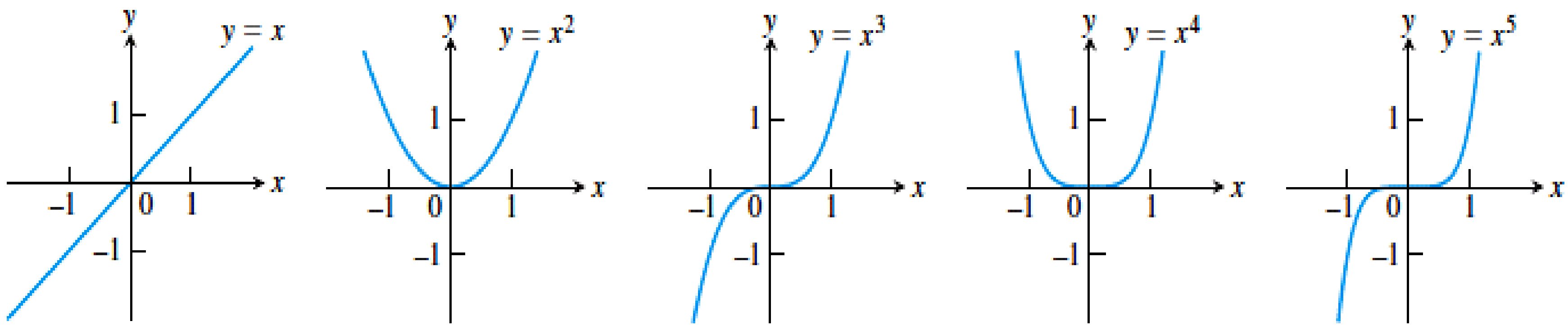
”A Constant functions has slope $m = 0$, Figure shows.



2. Power Functions: A function f(x) = x^a, where *a* is a constant, is called a **power function**. There are several important cases to consider.

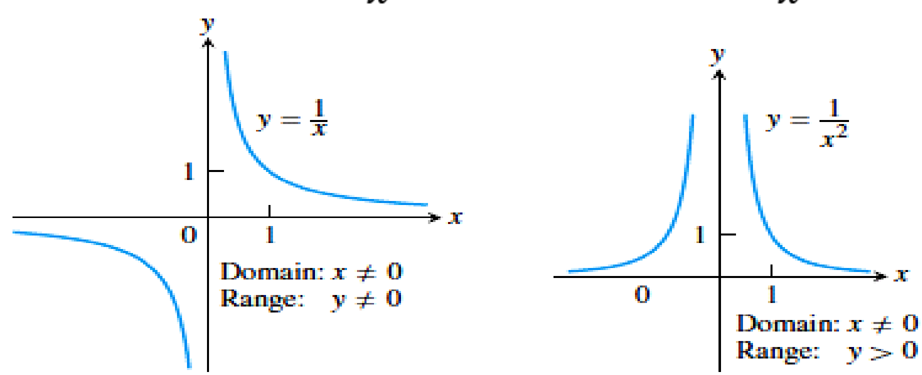
a. *n*, a positive integer.

The graphs of f(*x*) = x^n, for n = 1,2, 3, 4, 5, defined for $-\infty < x < \infty$. are displayed in Figure



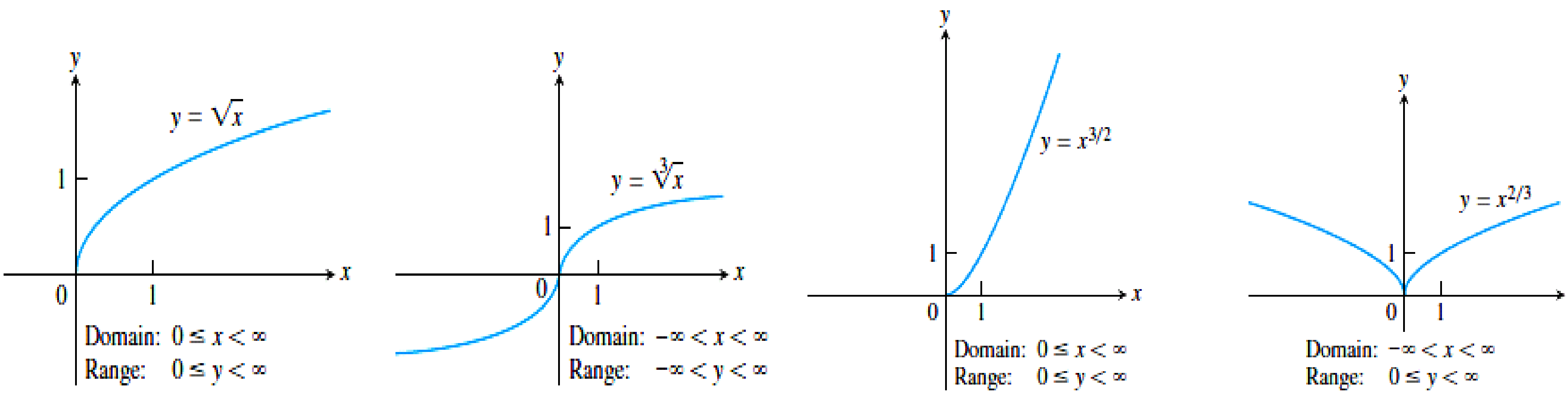
b. *n* = -1 or *n* = -2.

The graphs of the functions f(*x*) = $x^{-1} = \frac{1}{x}$ and *g*(*x*) = $x^{-2} = \frac{1}{x^2}$ are shown in Figure.



c. $n = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$, and $\frac{2}{3}$.

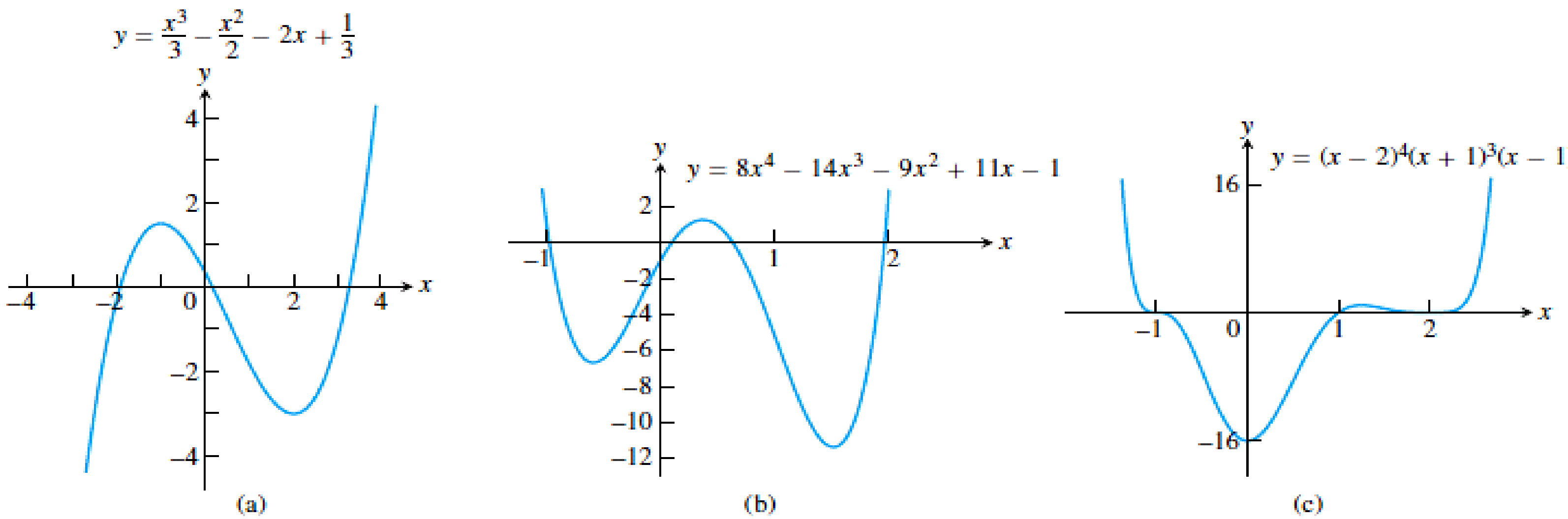
The function $f(x) = x^{\frac{1}{2}} = \sqrt{x}$, $g(x) = x^{\frac{1}{3}} = \sqrt[3]{x}$, $y = x^{\frac{3}{2}} = (x^{\frac{1}{2}})^3$, $y = x^{\frac{2}{3}} = (x^{\frac{1}{3}})^2$ The domain of the square root function is $[0,\infty)$, but the cube root function is defined for all real x. Their graphs are displayed in Figure.



3. Polynomials: A function p is a polynomial if

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

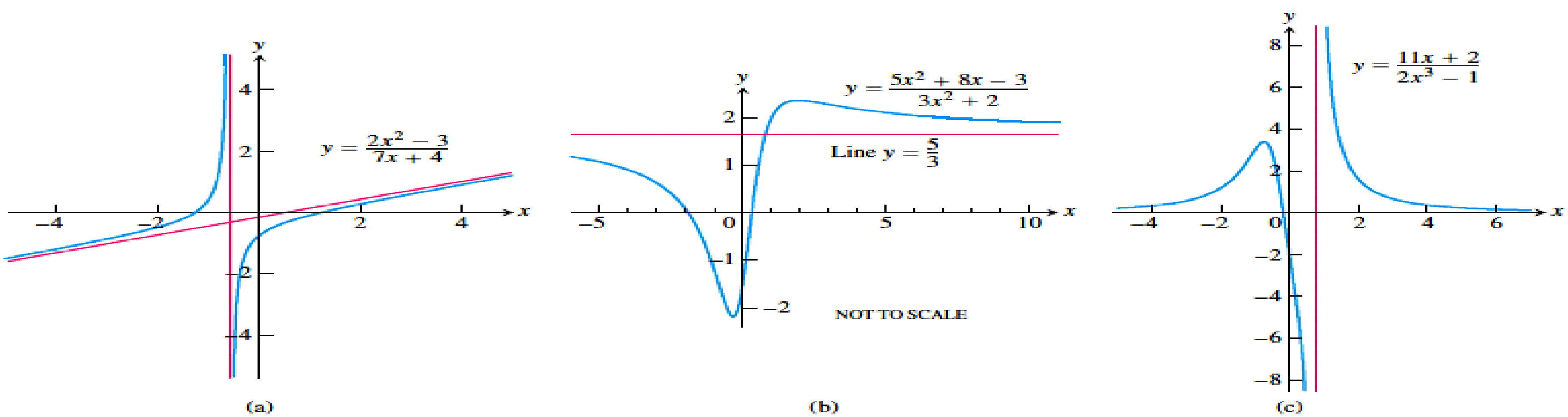
- All polynomials have domain $(-\infty, \infty)$. If the leading coefficient $a_n \neq 0$ and $n > 0$ then n is called the **degree** of the polynomial. Figure shows the graphs of three polynomials.



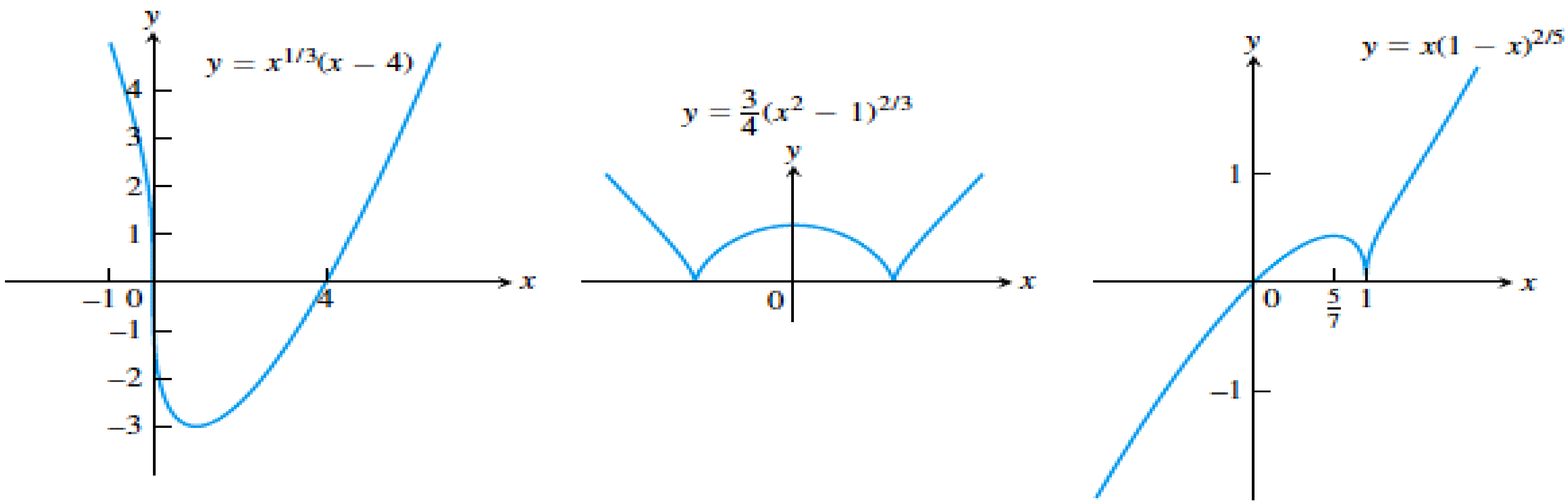
4. Rational Functions: A rational function is a ratio of two polynomials:

$f(x) = \dfrac{p(x)}{q(x)}$,The domain of a rational function is the set of all real x for which $q(x) \neq 0$.

For example, the function is shown in Figure.

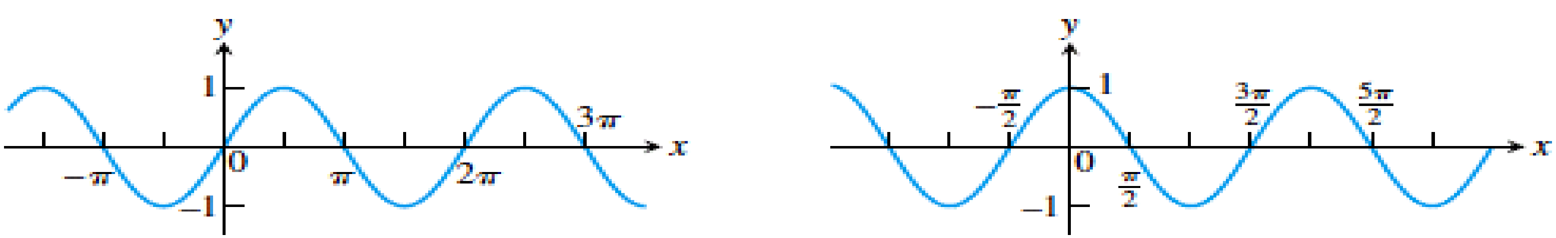


5. Algebraic Functions: An **algebraic function** is a function constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division, and taking roots). Rational functions are special cases of algebraic functions. Figure displays the graphs of three algebraic functions.



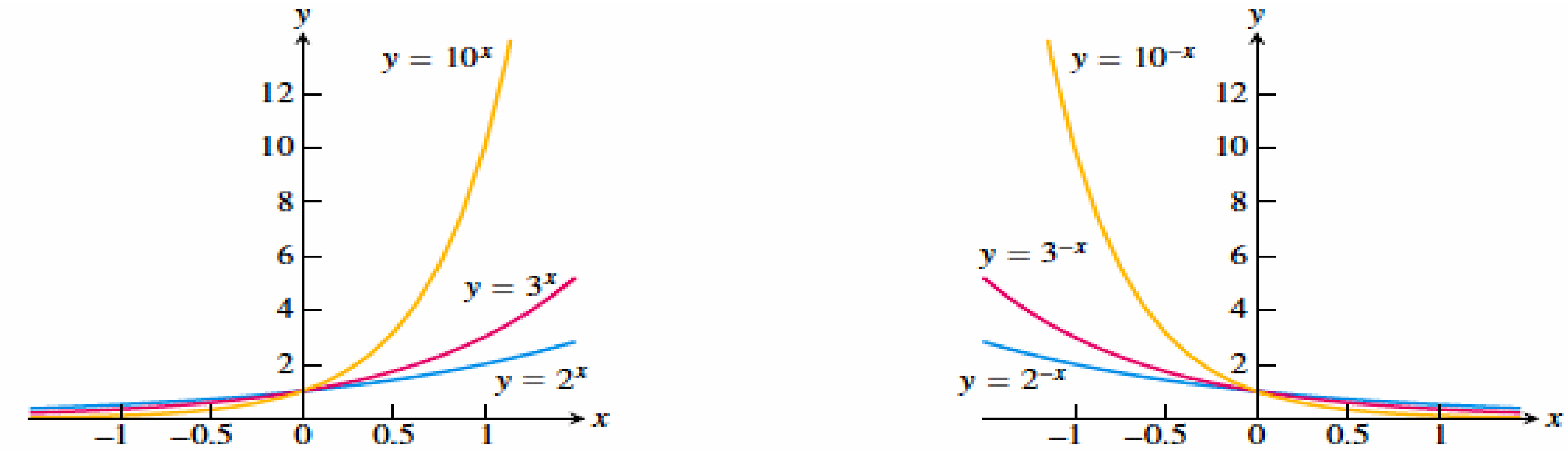
(a) (b) (c)

6. Trigonometric Functions: The graphs trigonometric functions of the sine and cosine functions are shown in Figure.



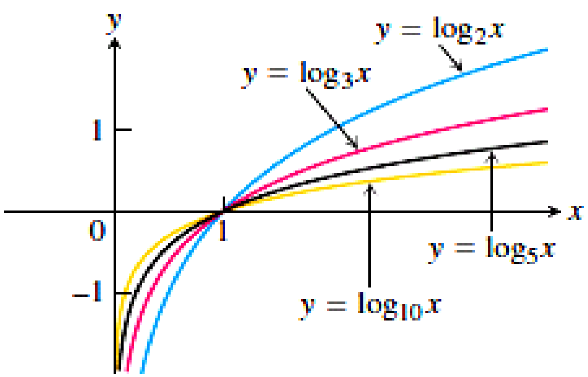
(a) $f(x) = \sin x$ (b) $f(x) = \cos x$

7. Exponential Functions: Functions of the form f(*x*) = a^x where the base is $a > 0$ is a positive constant and $a \neq 1$, are called exponential functions. All exponential functions have domain $(-\infty, \infty)$ and range $(0, \infty)$. The graphs of some exponential functions are shown in Figure.



(a) $y = 2^x, y = 3^x, y = 10^x$ (b) $y = 2^{-x}, y = 3^{-x}, y = 10^{-x}$

8. Logarithmic Functions: These are the functionsf(*x*) = log_ax, where the base $a \neq 1$ is a positive constant. Figure shows the graphs of four logarithmic functions with various bases. In each case the domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.



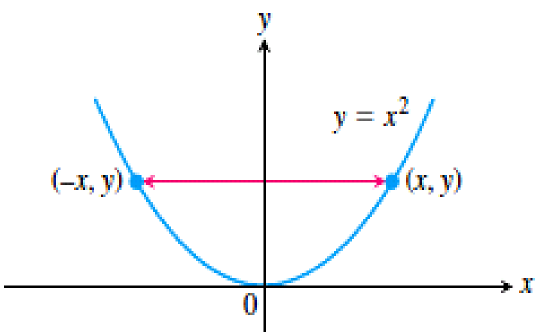
•Even Functions and Odd Functions: Symmetry

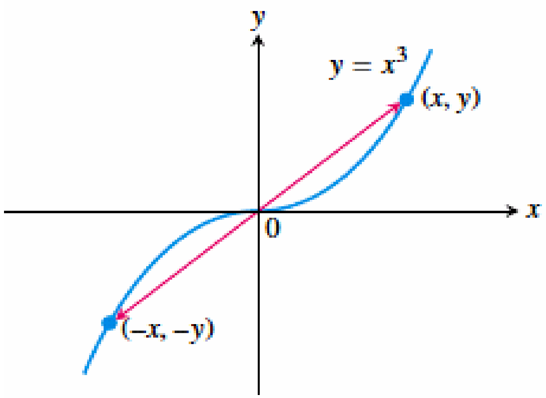
A function $y = f(x)$ is an

even function of *x* if $f(-x) = f(x)$,

odd function of *x* if $f(-x) = -f(x)$,

for every x in the function's domain.





”A function f is even if and only if the graph off is **symmetric about the *y*-axis**.

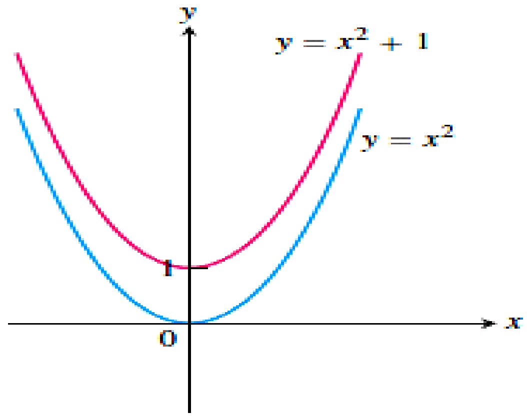
”The graph of an odd function is **symmetric about the origin**.

EXAMPLE 6: Say whether the function is Even, Odd, Functions. or neither.

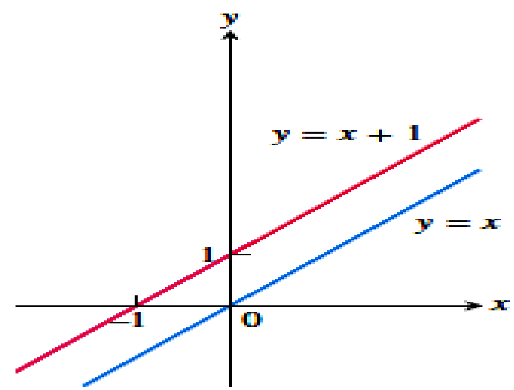
1. f(x) = x^2 2. f(x) = x^2 + 1 3. f(x) = x 4. f(x) = x + 1

Solution:

1. f(x) = x^2 → Even function $(-x)^2 = x^2$ for all x; symmetry about y-axis.
2. f(x) = x^2 + 1 → Even function $(-x)^2 + 1 = x^2 + 1$: for all x; symmetry about y-axis.



3. f(x) = x → Odd function(-x) = − x for all x; symmetry about the origin.
4. f(x) = x + 1 → Not odd: f(-x) = - x + 1, but –f(x) = -x - 1. The two are not equal.
   Not even: (-x) + 1 $\neq$ x + 1 for all $x \neq 0$



EXERCISES 2.3 Say whether the function is even, odd, or neither. Give reasons for your answer.

18. $f(x) = x^{-5}$

19. $g(x) = x^4 + x^2 - 1$

20. $h(t) = | t^3 |$

21. $y = \frac{x^4+1}{x^3-2x}$

22. $y = \sqrt{x^4 - 1}$

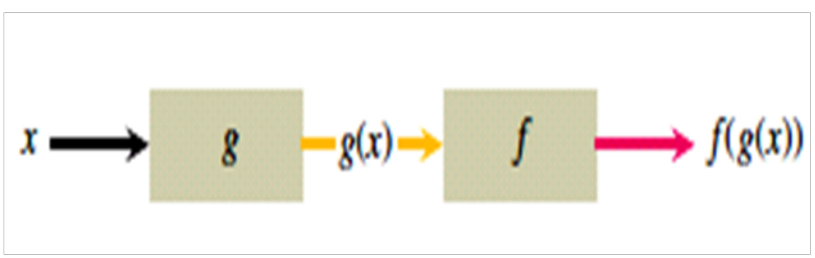
23. $y = x^5 - x^3 - x$

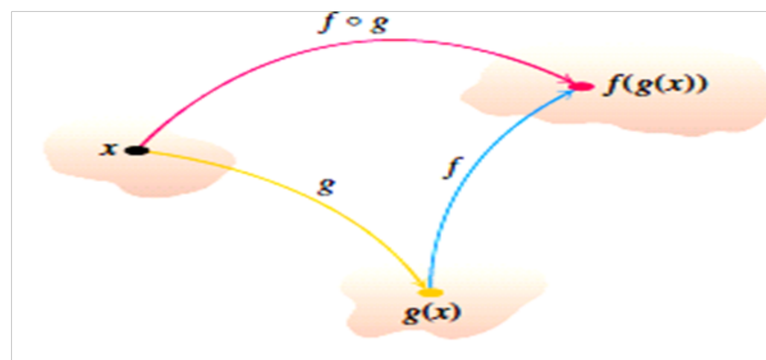
- **Composite Functions**

If f and *g* are functions, the **composite** function $f \circ g$ ("f composed with *g*") is defined by

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ consists of the numbers *x* in the domain of *g* for which *g*(*x*) lies in the domain of f. To find $(f \circ g)(x)$, *first* find g(x) and *second* find *f(g(x))*.





Arrow diagram for $f \circ g$.

EXAMPLE 7: If $f(x) = \sqrt{x}$ and $g(x) = x+1$, find

(a) $(f \circ g)(x)$ ***(b)*** $(g \circ f)(x)$ ***(c)*** $(f \circ f)(x)$ ***(e)*** $(g \circ g)(X)$.

Solution:

Composite	**Domain**
(a) $(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x+1}$	$[-1, \infty)$
(b) $(g \circ f)(x) = g(f(x)) = f(x) + 1 = \sqrt{x} + 1$	$[0, \infty)$
(c) $(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{\frac{1}{4}}$	$[0, \infty)$
(d) $(g \circ g)(x) = g(g(x)) = g(x) + 1 = (x + 1) + 1 = x + 2$	$(-\infty, \infty)$

"Notice: that if $f(x) = x^2$ and $g(x) = \sqrt{x}$ then $(f \circ g)(x) = (\sqrt{x})^2 = x$. However, the domain of $f \circ g$ is $[0, \infty)$, not $(-\infty, \infty)$

- ## Shifting a Graph of a Function

Vertical Shifts

y = f(x) + k Shifts the graph of *f* up *k* units *if* $k > 0$

Shifts it down |*k*| units if $k < 0$

Horizontal Shifts

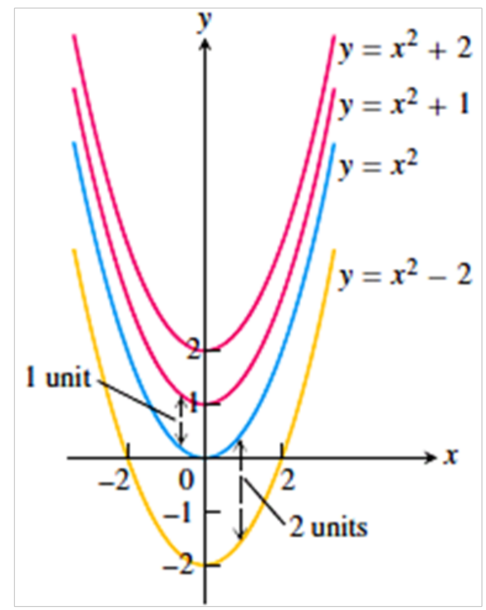
y = f(x + h) Shifts the graph of *f* left *h* units if $h > 0$

Shifts it right |*h* | units if $h < 0$

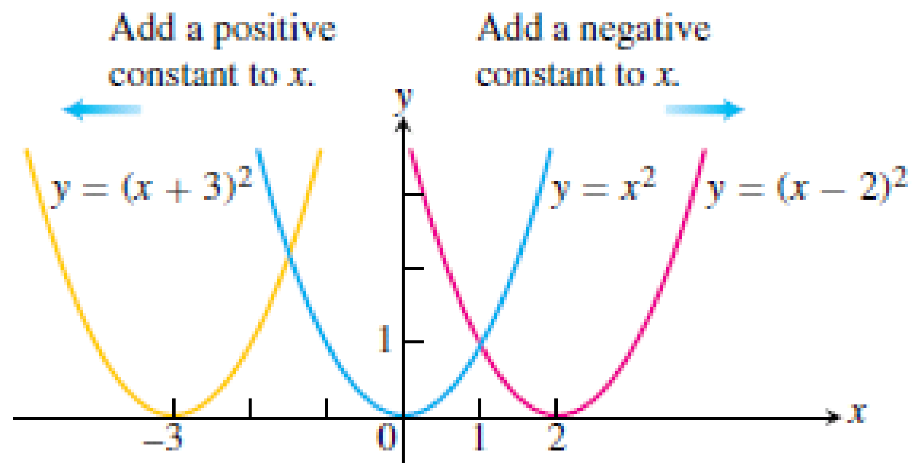
EXAMPLE 8 Shifting a Graph

(a) Adding 1 to the right-hand side of the formula $y = x^2$ to get $y = x^2 + 1$ shifts the graph up 1 unit (shows in Figure).

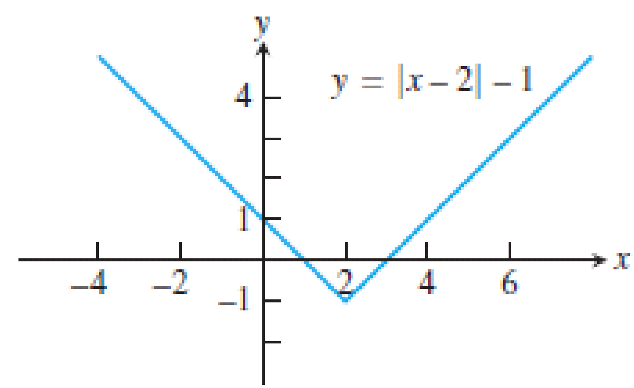
(b) Adding −2 to the right-hand side of the formula $y = x^2$ to get $y = x^2 - 2$ shifts the graph down 2 units (shows in Figure).



(c) Adding 3 to x in $y = x^2$ to get $y = (x + 3)^2$ shifts the graph 3 units to the left (shows in Figure).



(d) Adding −2 to x in $y = |x|$, and then adding −1 to the result, gives $y = | x - 2| - 1$ and shifts the graph 2 units to the right and 1 unit down (shows in Figure).



EXERCISES 2.3

1. If $f(x) = x+5$ and $g(x) = x^2 - 3$, find the following.

   a. $f(g(0))$ b. $g(f(X))$ c. $f(f(-5))$

2. If $f(x) = x - 1$ and $g(x) = \frac{1}{x+1}$, find the following.

   a. $f(g(1/2))$ b. $g(f(x))$ c. $f(f(2))$

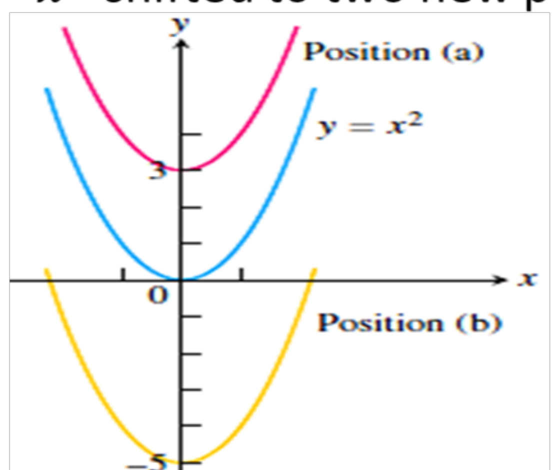
3. If $f(x) = \sqrt{x}$ and $g(x) = \frac{x}{4}$ and $h(x) = 4x - 8$, find formulas for the following.

   a. $h(g(f(x)))$ b. $g(f(h(x)))$ c. $f(h(g(x)))$

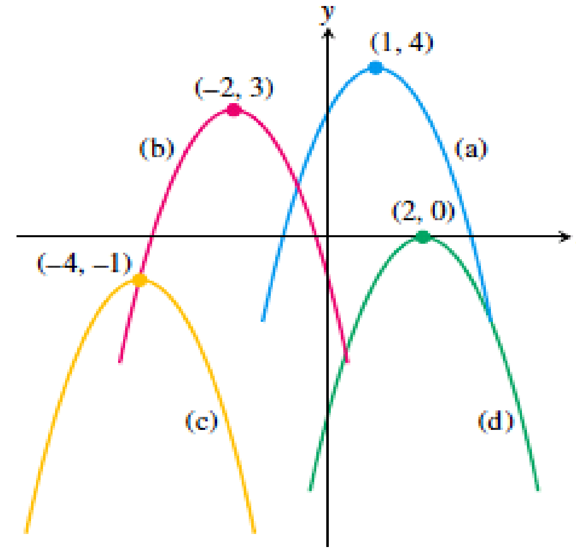
4. write a formula for $f \circ g$ and $g \circ f$ and find the domain and range of each.

   $F(x) = \sqrt{x+1}$, $g(x) = \frac{1}{x}$

5. figure shows the graph of $y = -x^2$ shifted to two new positions. Write equations for the new graphs.



6 . figure shows the graph of $y = -x^2$ shifted to four new positions. Write an equation for each new graph.



7 Match the equations listed in parts (a)–(d) to the graphs in the accompanying figure.

a. $y = (x-1)^2 - 1$ b. $y = (x-2)^2 + 2$

c. $y = (x+2)^2 + 2$ c. $y = (x+3)^2 - 2$