MATHEMATICS I

FIRST SEMESTER

Lec. 05

Limits And Continuity

Outlines

- Limit of a Function
- Calculating Limits Using the Limit Laws
- One-Sided Limits and Limits at Infinity
  - One-Sided Limits
  - Limits Involving **(sin θ)/θ**
  - Finite Limits as $x \to \pm\infty$
  - Continuous Functions

Limit of a Function

Let $f(x)$ be defined on an open interval about x_0 except possibly at itself. We say that the limit of $f(x)$ as x approaches x_0 is the number L, and write

$$\lim_{x \to x_0} f(x) = L,$$

Example 1: The Identity and Constant Functions Have Limits at Every Point

(a) If f is the **identity function** $f(x) = x$, then for any value of x_0

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} x = x_0.$$

$$\lim_{x \to 2} (4) = 4 \qquad \lim_{x \to -13} (4) = 4$$

(b) If f is the **constant function** $f(x) = k$ (function with the constant value k), then for any value of x_0

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} k = k.$$

For instance,

$$\lim_{x \to 3} x = 3 \quad \text{and} \quad \lim_{x \to -7} (4) = \lim_{x \to 2} (4) = 4.$$

- **Calculating Limits Using the Limit Laws**

The next theorem tells how to calculate limits of functions

THEOREM 1 **Limit Laws**

If L, M, c and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule*: $\lim_{x \to c}(f(x) + g(x)) = L + M$

The limit of the sum of two functions is the sum of their limits.

2. *Difference Rule*: $\lim_{x \to c}(f(x) - g(x)) = L - M$

The limit of the difference of two functions is the difference of their limits.

3. *Product Rule*: $\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$

The limit of a product of two functions is the product of their limits.

4. *Constant Multiple Rule:* $\lim_{x \to c}(k \cdot f(x)) = k \cdot L$

The limit of a constant times a function is the constant times the limit of the function.

5. *Quotient Rule*: $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$

The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

6. *Power Rule*: If r and s are integers with no common factor and $s \neq 0$, then

$$\lim_{x \to c}(f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

EXAMPLE 2: Using the Limit Laws to find the following limits.

(a) $\lim_{x \to c} (x^3 + 4x^2 - 3)$ (b) $\lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5}$ (c) $\lim_{x \to -2} \sqrt{4x^2 - 3}$

Solution

(a) $\lim_{x \to c} (x^3 + 4x^2 - 3) = \lim_{x \to c} x^3 + \lim_{x \to c} 4x^2 - \lim_{x \to c} 3$ Sum and Difference Rules

$= c^3 + 4c^2 - 3$ Product and Multiple Rules

(b) $\lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x \to c} (x^4 + x^2 - 1)}{\lim_{x \to c} (x^2 + 5)}$ Quotient Rule

$= \frac{\lim_{x \to c} x^4 + \lim_{x \to c} x^2 - \lim_{x \to c} 1}{\lim_{x \to c} x^2 + \lim_{x \to c} 5}$ Sum and Difference Rules

$= \frac{c^4 + c^2 - 1}{c^2 + 5}$ Power or Product Rule

(c) $\lim_{x \to -2} \sqrt{4x^2 - 3} = \sqrt{\lim_{x \to -2} (4x^2 - 3)}$ Power Rule with $r/s = \frac{1}{2}$

$= \sqrt{\lim_{x \to -2} 4x^2 - \lim_{x \to -2} 3}$ Difference Rule

$= \sqrt{4(-2)^2 - 3}$ Product and Multiple Rules

$= \sqrt{16 - 3}$

$= \sqrt{13}$

THEOREM 2 Limits of Polynomials Can Be Found by Substitution

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

THEOREM 3 Limits of Rational Functions Can Be Found by Substitution If the Limit of the Denominator Is Not Zero

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Example 3: Evaluate $\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x}$.

Solution : We cannot substitute because it makes the denominator zero.

$$\frac{x^2 + x - 2}{x^2 - x} = \frac{(x-1)(x+2)}{x(x-1)} = \frac{x+2}{x}, \qquad \text{if } x \neq 1.$$

Using the simpler fraction, we find the limit of these values as $x \to 1$ by substitution:

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \to 1} \frac{x+2}{x} = \frac{1+2}{1} = 3.$$

EXAMPLE 4 Creating and Canceling a Common Factor

Evaluate

$$\lim_{x \to 0} \frac{\sqrt{x^2 + 100} - 10}{x^2}.$$

Solution: We cannot substitute $x = 0$, and the numerator and denominator have no obvious common factors. We can create a common factor by multiplying both numerator and denominator by the expression $\sqrt{x^2+100}+10$,

$$\frac{\sqrt{x^2+100}-10}{x^2} = \frac{\sqrt{x^2+100}-10}{x^2} \cdot \frac{\sqrt{x^2+100}+10}{\sqrt{x^2+100}+10}$$

$$= \frac{x^2+100-100}{x^2\left(\sqrt{x^2+100}+10\right)}$$

$$= \frac{x^2}{x^2\left(\sqrt{x^2+100}+10\right)} \qquad \text{Common factor } x^2$$

$$= \frac{1}{\sqrt{x^2+100}+10}. \qquad \text{Cancel } x^2 \text{ for } x \neq 0$$

Therefore,

$$\lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2} = \lim_{x\to 0}\frac{1}{\sqrt{x^2+100}+10}$$

$$= \frac{1}{\sqrt{0^2+100}+10} \qquad \text{Denominator not 0 at } x = 0\text{; substitute}$$

$$= \frac{1}{20} = 0.05.$$

EXERCISES 5.1:

"Find the limits in the following Exercises

1. $\lim_{x\to -2}(x^3 - 2x^2 + 4x + 8)$
2. $\lim_{s\to 2/3} 3s(2s - 1)$
3. $\lim_{y\to -3}(5 - y)^{4/3}$
4. $\lim_{y\to -3}(5 - y)^{4/3}$
5. $\lim_{y\to 2} \frac{y + 2}{y^2 + 5y + 6}$
6. $\lim_{x\to -4}(x + 3)^{1984}$
7. $\lim_{x\to -5} \frac{x^2 + 3x - 10}{x + 5}$
8. $\lim_{x\to 4} \frac{4x - x^2}{2 - \sqrt{x}}$
9. $\lim_{x\to 2} \frac{\sqrt{x^2 + 12} - 4}{x - 2}$
10. $\lim_{x\to 1} \frac{1 - \sqrt{x}}{1 - x}$
11. $\lim_{x\to 0} \frac{(2 + x)^3 - 8}{x}$
12. $\lim_{x\to 0} \frac{\frac{1}{2 + x} - \frac{1}{2}}{x}$

"Using Limit Rules to solve the following Exercises .

1. Suppose $\lim_{x\to 0} f(x) = 1$ and $\lim_{x\to 0} g(x) = -5$. Name the rules in Theorem 1

a. $\lim_{x\to 0} \frac{2f(x) - g(x)}{(f(x) + 7)^{2/3}}$

2. Let $\lim_{x\to 1} h(x) = 5$, $\lim_{x\to 1} p(x) = 1$, and $\lim_{x\to 1} r(x) = 2$. Name the rules in Theorem 1

a. $\lim_{x\to 1} \frac{\sqrt{5h(x)}}{p(x)(4 - r(x))}$

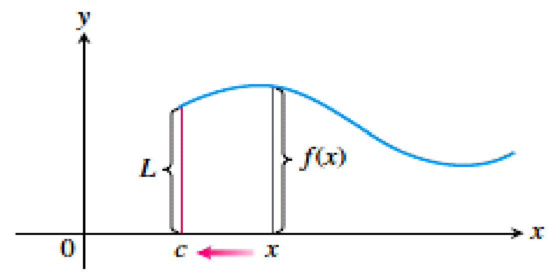
One-Sided Limits and Limits at Infinity

In this section we extend the limit concept to *one-sided limits*, which are limits as x approaches the number x_0 from the left-hand side (where $x < x_0$) or the right-hand side ($x > x_0$) only, and other functions with limit behavior as $x \to \pm\infty$.

” One-Sided Limits:

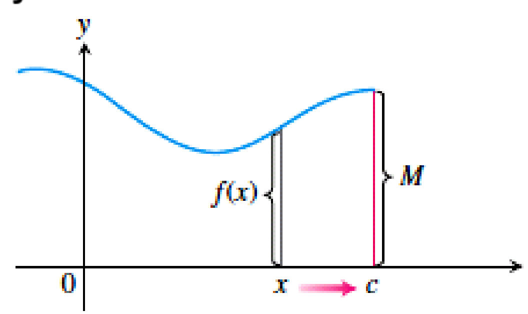
a. if f(x) is defined on an interval (c, b), where $c < b$ and approaches arbitrarily close to L as x approaches c from within that interval, then f has **right-hand limit** L at c. We write, $\lim_{x \to c^+} f(x) = L.$

The symbol “$x \to c+$” means that we consider only values of x greater than c.



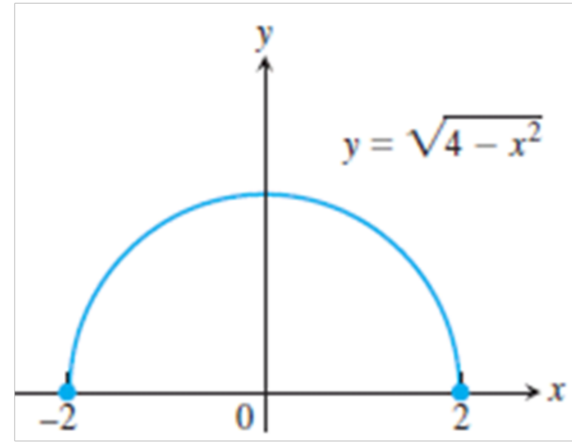
b. if f(x) is defined on an interval (a, c), where $a < c$ and approaches arbitrarily close to M as x approaches c from within that interval, then f has **left-hand limit** M at c. We write, $\lim_{x \to c^-} f(x) = M.$

The symbol “$x \to c-$” means that we consider only x values less than c.



EXAMPLE 5 One-Sided Limits for a Semicircle shown in figure.

The domain of $f(x) = \sqrt{4 - x^2}$ is [-2,2] . We have,



$$\lim_{x \to -2^+} \sqrt{4 - x^2} = 0 \quad \text{and} \quad \lim_{x \to 2^-} \sqrt{4 - x^2} = 0.$$

The function does not have a left-hand limit at $x = -2$ or a right-hand limit at $x = 2$. It does not have ordinary two-sided limits at either -2 or 2. (the function is one-side limits)

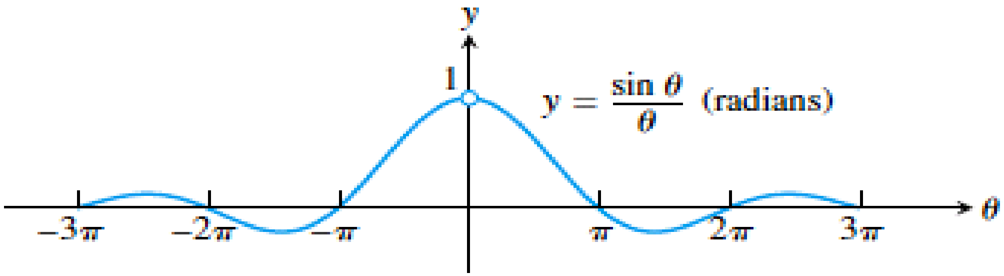
THEOREM 6

A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

•Limits Involving $(\sin\theta)/\theta$

A central fact about $(\sin\theta)/\theta$ is that in radian measure its limit as $\theta \to 0$ is 1. We can see this in Figure,



THEOREM 7

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \qquad (\theta \text{ in radians}) \qquad (1)$$

EXAMPLE 6: Using $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ Show that,

a. $\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$

b. $\lim_{x \to 0} \frac{\sin 2x}{5x} = \frac{2}{5}$.

Solution:

(a) Using the half-angle formula,

Note. When $sin^2\boldsymbol{\theta} = \frac{1-cos2\boldsymbol{\theta}}{2} \rightarrow 2\ sin^2\boldsymbol{\theta} = 1 - cos2\boldsymbol{\theta} \rightarrow \cos 2\boldsymbol{\theta} = 1 - 2\ sin^2\boldsymbol{\theta}$

Let, $2\boldsymbol{\theta} = h \rightarrow \boldsymbol{\theta} = \frac{h}{2}$, than $\cosh = 1 - 2\ sin^2 \frac{h}{2}$.

The , $\lim_{h\to 0} \frac{\cosh - 1}{h} = \lim_{h\to 0} \frac{1-2sin^2\ (h/_2) - 1}{h} = \lim_{h\to 0} - \frac{2sin^2\ (h/_2)}{h}$ divided by 2 obtain =

$- \lim_{h\to 0} \frac{sin^2\ (h/_2)}{h/2} = - \lim_{h\to 0} \frac{\sin(h/_2) * \sin(h/_2)}{h/2} = -(1) \times \lim_{h\to 0} \sin(h/_2)$

$= -(1) \times \sin(0/2) = -(1) \times (0) = 0$

(b) We need a 2*x* in the denominator,not a 5*x*. We produce it by multiplying numerator and denominator by $\frac{5}{2}$:

$$\lim_{x\to 0} \frac{\sin 2x}{5x} = \lim_{x\to 0} \frac{(2/5) \cdot \sin 2x}{(2/5) \cdot 5x}$$

$$= \frac{2}{5} \lim_{x\to 0} \frac{\sin 2x}{2x}$$

$$= \frac{2}{5}(1) = \frac{2}{5}$$

• Finite Limits as $x \to \pm\infty$

1. We say that $f(x)$ has the **limit L as x approaches infinity** and write

$$\lim_{x \to \infty} f(x) = L$$

2. We say that $f(x)$ has the **limit L as x approaches minus infinity** and write

$$\lim_{x \to -\infty} f(x) = L$$

THEOREM 8 Limit Laws as $x \to \pm\infty$

If L, M, and k, are real numbers and

$$\lim_{x \to \pm\infty} f(x) = L \quad \text{and} \quad \lim_{x \to \pm\infty} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x \to \pm\infty} (f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x \to \pm\infty} (f(x) - g(x)) = L - M$
3. *Product Rule:* $\lim_{x \to \pm\infty} (f(x) \cdot g(x)) = L \cdot M$
4. *Constant Multiple Rule:* $\lim_{x \to \pm\infty} (k \cdot f(x)) = k \cdot L$
5. *Quotient Rule:* $\lim_{x \to \pm\infty} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* If r and s are integers with no common factors, $s \neq 0$, then

$$\lim_{x \to \pm\infty} (f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

EXAMPLE 7 Find the following limits of a functions as $x \to \pm\infty$,

(a) $$\lim_{x\to\infty}\left(5+\frac{1}{x}\right) = \lim_{x\to\infty} 5 + \lim_{x\to\infty}\frac{1}{x}$$
$$= 5 + 0 = 5$$

(b) $$\lim_{x\to-\infty}\frac{\pi\sqrt{3}}{x^2} = \lim_{x\to-\infty}\pi\sqrt{3}\cdot\frac{1}{x}\cdot\frac{1}{x}$$
$$= \lim_{x\to-\infty}\pi\sqrt{3}\cdot\lim_{x\to-\infty}\frac{1}{x}\cdot\lim_{x\to-\infty}\frac{1}{x}$$
$$= \pi\sqrt{3}\cdot 0\cdot 0 = 0$$

(c)
$$\lim_{x\to\infty}\frac{5x^2+8x-3}{3x^2+2} = \lim_{x\to\infty}\frac{5+(8/x)-(3/x^2)}{3+(2/x^2)}$$
$$= \frac{5+0-0}{3+0} = \frac{5}{3}$$

Divide numerator and denominator by x^2.

(e)
$$\lim_{x\to-\infty}\frac{11x+2}{2x^3-1} = \lim_{x\to-\infty}\frac{(11/x^2)+(2/x^3)}{2-(1/x^3)}$$
$$= \frac{0+0}{2-0} = 0$$

Divide numerator and denominator by x^3.

•Continuous Functions

DEFINITION Continuous at a Point

Interior point: A function $y = f(x)$ is **continuous at an interior point c** of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

Endpoint: A function $y = f(x)$ is **continuous at a left endpoint a** or is **continuous at a right endpoint b** of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively.}$$

If a function f is not continuous at a point c, we say that f is **discontinuous** at c and c is a **point of discontinuity** of f. Note that c need not be in the domain of f.

A function f is **right-continuous (continuous from the right)** at a point $x = c$ in its domain if $\lim_{x \to c^+} f(x) = f(c)$. It is **left-continuous (continuous from the left)** at c if

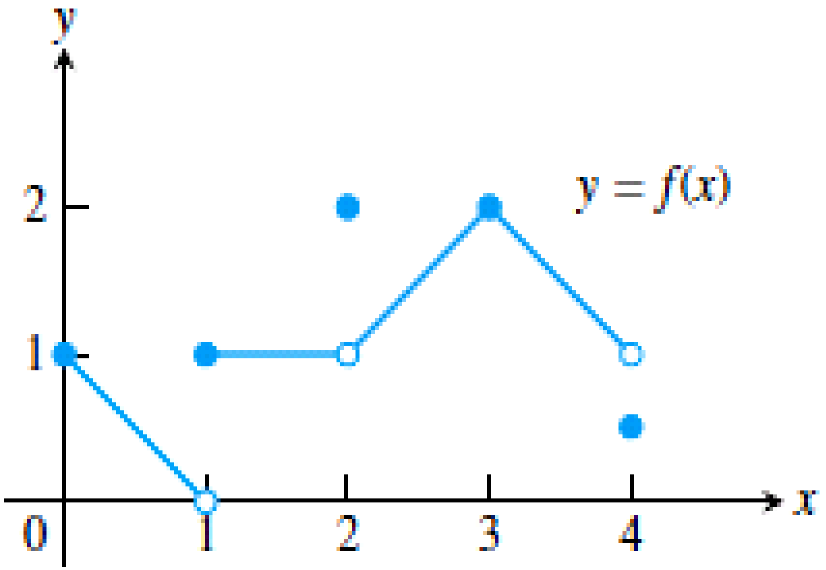
$\lim_{x \to c^-} f(x) = f(c)$.

Example 8:

Find the points at which the function f in Figure is continuous and the points at which f is discontinuous.

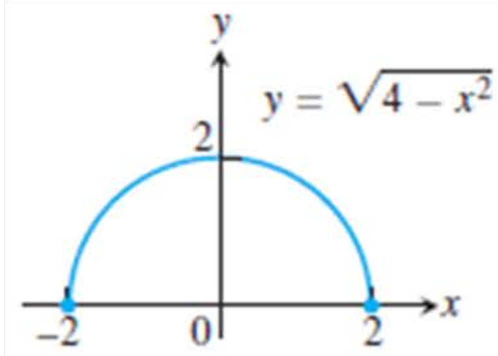
Solution:

The function f is continuous at every point in its domain $[0, 4]$ except at $x = 1$, $x = 2$, and $x = 4$.



EXAMPLE 9 : A Function Continuous

The function $f(x) = \sqrt{4 - x^2}$ is continuous at every point of its domain, [-2, 2] , (Figure), including $x = -2$, where f is right-continuous, and $x = 2$, where f is left-continuous .



We summarize continuity at a point in the form of a test.

Continuity Test

A function $f(x)$ is continuous at $x = c$ if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f)
2. $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$)
3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value)

Example 10:

$$f(x) = \begin{cases} 3+x & x \le 1 \\ 3-x & x > 1 \end{cases}$$

$$f(1) = 3 + 1 = 4$$

$$\lim_{x \to 1^-} 3 + x = 3 + 1 = 4$$

$$\lim_{x \to 1^+} 3 - x = 3 - 1 = 2 \qquad \therefore \lim_{x \to 1^-} 3 + x \neq \lim_{x \to 1^+} 3 - x$$

$\therefore f(x)$ *discontinuous at* $x = 1$.

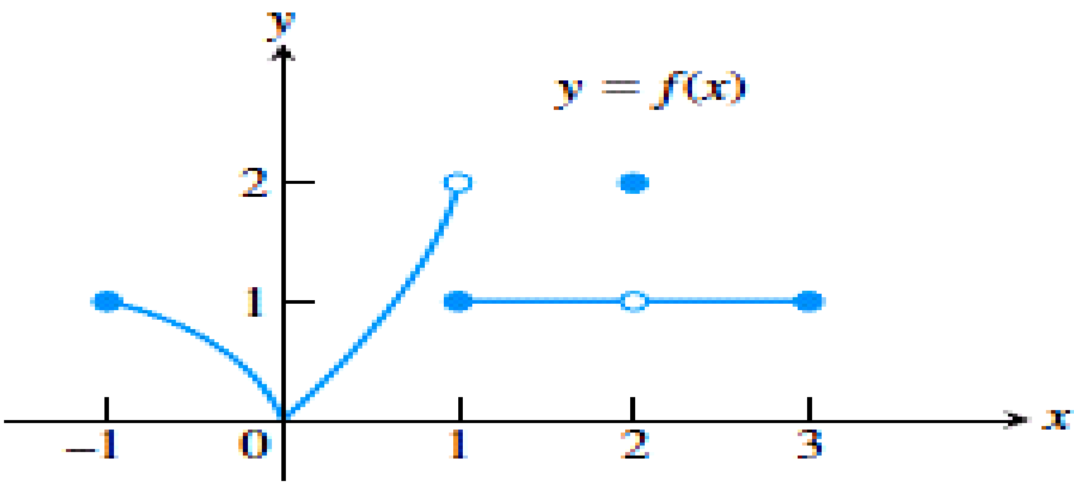
Example 11:

$$f(x) = \begin{cases} \frac{1}{x-2} & x \neq 2 \\ 3 & x = 2 \end{cases}$$

$f(2) = 3$ & $\lim_{x \to 2} \frac{1}{x-2} = \frac{1}{0} = \infty$ $\quad \therefore$ *no limit* , $f(x)$ *discontinuous*.

EXERCISES 5.2:

1 Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



a. $\lim_{x \to -1^+} f(x) = 1$

b. $\lim_{x \to 2} f(x)$ does not exist.

c. $\lim_{x \to 2} f(x) = 2$

d. $\lim_{x \to 1^-} f(x) = 2$

e. $\lim_{x \to 1^+} f(x) = 1$

f. $\lim_{x \to 1} f(x)$ does not exist.

g. $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x)$

h. $\lim_{x \to c} f(x)$ exists at every c in the open interval $(-1, 1)$.

i. $\lim_{x \to c} f(x)$ exists at every c in the open interval $(1, 3)$.

j. $\lim_{x \to -1^-} f(x) = 0$

k. $\lim_{x \to 3^+} f(x)$ does not exist.

2. a. Graph $f(x) = \begin{cases} 1 - x^2, & x \neq 1 \\ 2, & x = 1. \end{cases}$

b. Find $\lim_{x \to 1^+} f(x)$ and $\lim_{x \to 1^-} f(x)$.

c. Does $\lim_{x \to 1} f(x)$ exist? If so, what is it? If not, why not?

3. Find One-Sided Limits Algebraically In Following Exercises .

a. $\lim_{x \to -0.5^-} \sqrt{\frac{x+2}{x+1}}$

b. $\lim_{x \to -2^+} \left(\frac{x}{x+1}\right)\left(\frac{2x+5}{x^2+x}\right)$

c. $\lim_{h \to 0^-} \frac{\sqrt{6} - \sqrt{5h^2 + 11h + 6}}{h}$

d. $\lim_{x \to 1^+} \frac{\sqrt{2x}\,(x-1)}{|x-1|}$

4. Using $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$, Find the limits in following Exercises.

1. $\lim_{t \to 0} \frac{\sin(1 - \cos t)}{1 - \cos t}$

2. $\lim_{x \to 0} \frac{x \csc 2x}{\cos 5x}$

3. $\lim_{x \to 0} \frac{x + x\cos x}{\sin x \cos x}$

5. Find the limit of each function **(a)** as $x \to \infty$ and **(b)** as $x \to -\infty$.

1. $g(x) = \frac{1}{8 - (5/x^2)}$

2. $h(x) = \frac{-5 + (7/x)}{3 - (1/x^2)}$

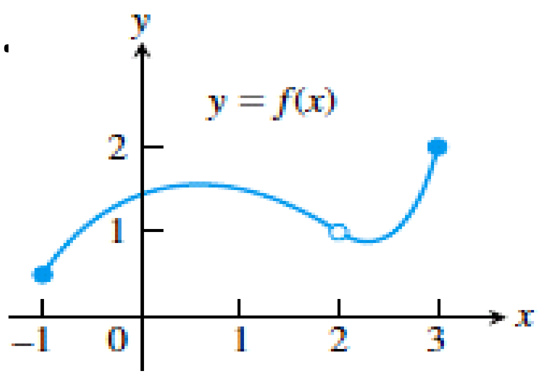
3. $h(x) = \frac{3 - (2/x)}{4 + (\sqrt{2}/x^2)}$

4. $f(x) = \dfrac{2x^3 + 7}{x^3 - x^2 + x + 7}$

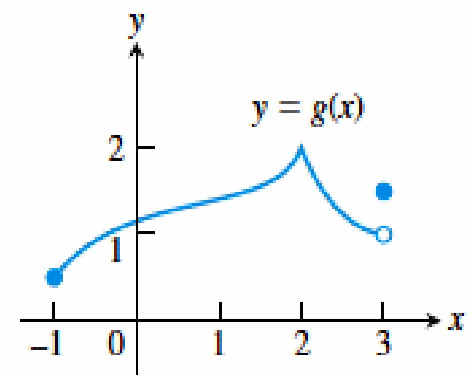
5. $f(x) = \dfrac{3x + 7}{x^2 - 2}$

6. In Exercises 1–3, say whether the function graphed is continuous on [-1 , 3].

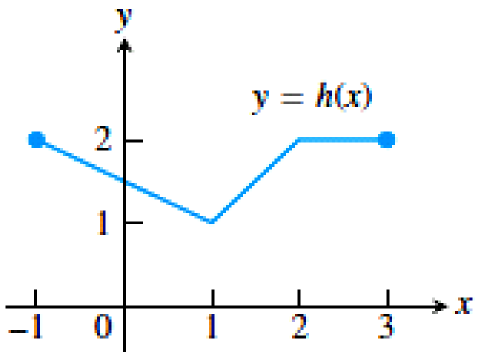
1.



2.



3.



8. At what points are the functions in Exercises 1–3 continuous?

1. $y = \dfrac{1}{(x+2)^2} + 4$

2. $y = \dfrac{x+3}{x^2 - 3x - 10}$

3. $y = \sqrt[4]{3x - 1}$

4. $y = (2 - x)^{1/5}$

9. Define $g(3)$ in a way that extends $g(x) = (x^2 - 9) / (x - 3)$ to be continuous at $x = 3$.

10. Define $g(4)$ in a way that extends $g(x) = (x^2 - 16) / (x^2 - 3x - 4)$ to be continuous at $X = 4$.

11. For what value of a is, $f(x) = \begin{cases} x^2 - 1, & x < 3 \\ 2ax, & x \geq 3 \end{cases}$, continuous at every x?